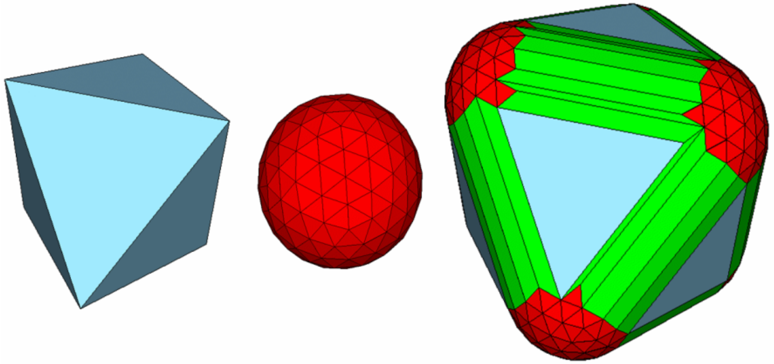
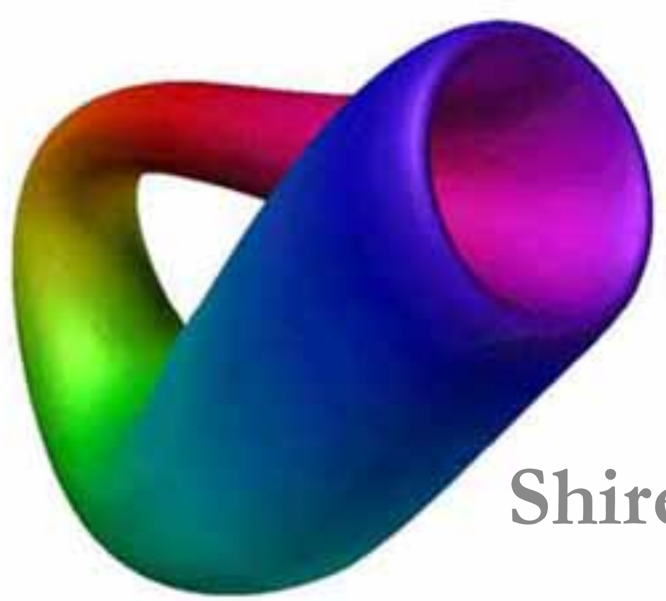
A Flavor of Topology





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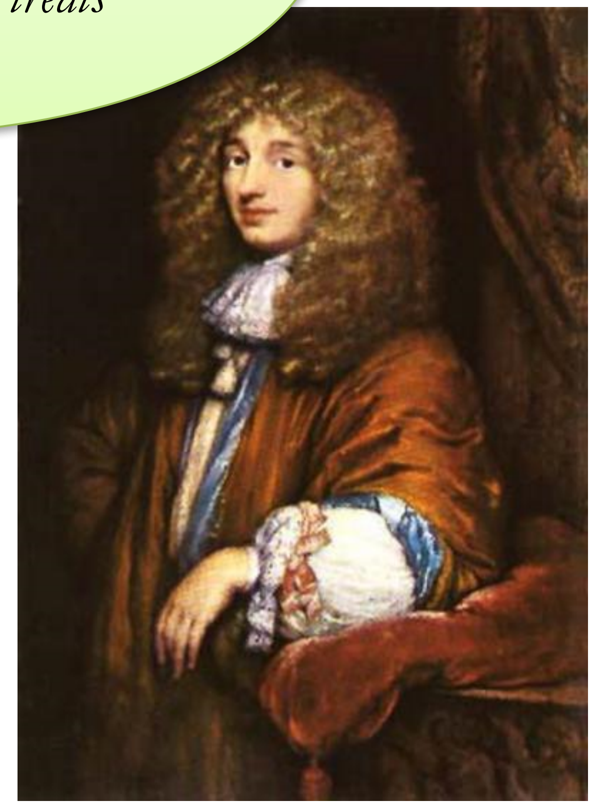
In 1670's …

"I believe that we need another analysis properly geometric or linear, which treats place directly the way that algebra treats magnitude."

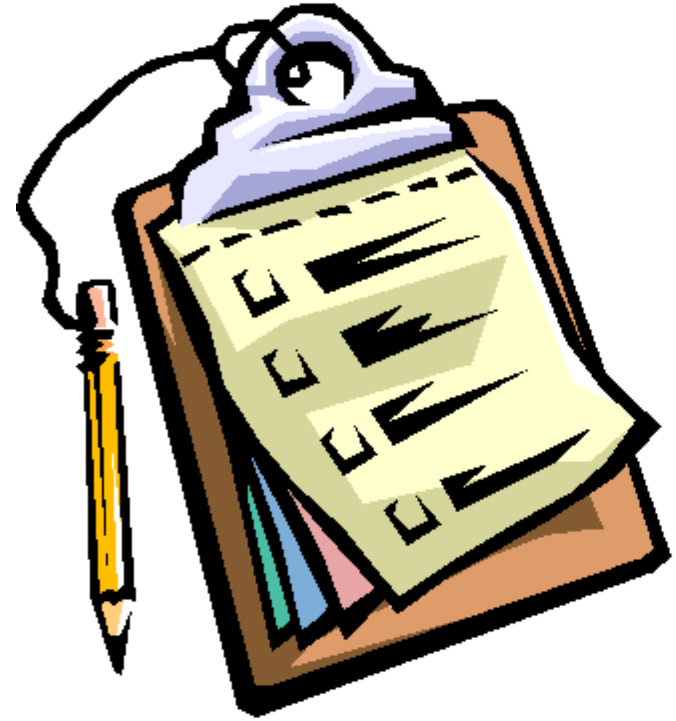
Gottfried W. Leibniz
German philosopher and mathematician



Christiaan Huygens
prominent Dutch mathematician, astronomer, physicist, and horologist

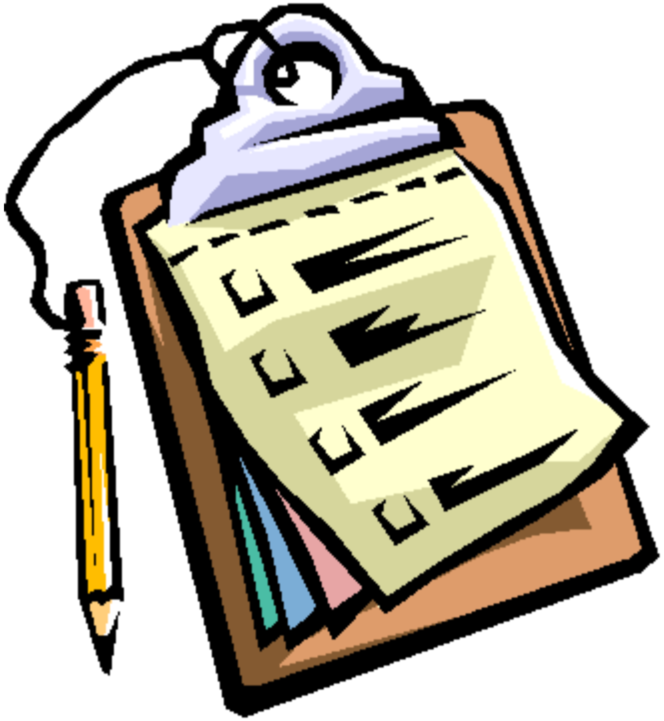
Leibniz was dreaming of a calculus of figures where one can do arithmetic with figures with the level of ease as with numbers.

Agenda



- Overview
- Functions
- Topological equivalence
- Point-set topology in R^n
- Point-set topology – more abstract level
- Surfaces

Agenda



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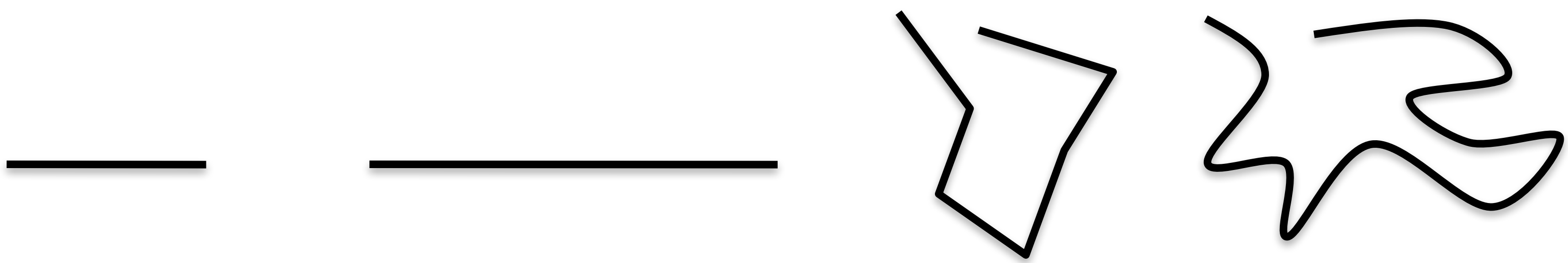
Overview

- **Topology** is a major branch of mathematics that benefits a lot from concepts of geometry, set theory and group theory, such as space, dimension, shape, transformations and others.
- In **geometry**,
  - We study primitive figures such as triangles, parallelograms, polygons and others;
  - We also study their geometry properties such as side lengths, angle measures and areas enclosed.
  - A step further is learning when figures are geometrically the same or congruent, i.e. having same geometrical properties.
  - Then we study functions such as rotations which preserve congruence, such function belong to a set of functions which perform rigid motion on the given figure, they are commonly denoted as isometries.

Overview

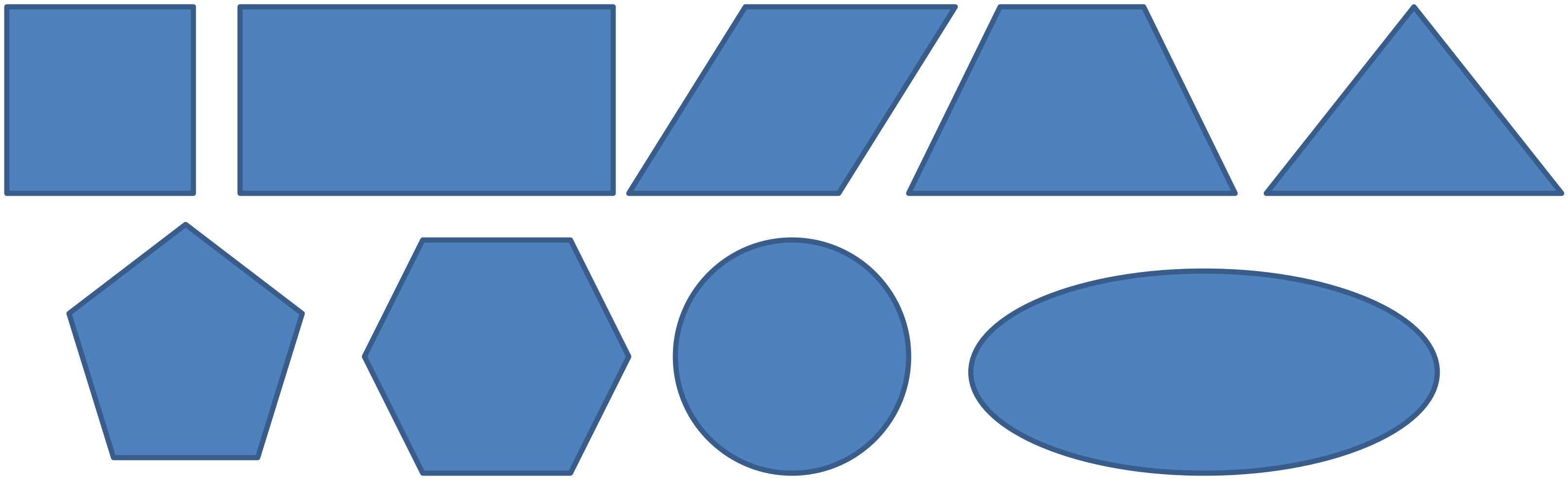
- In **topology**; We can ignore geometric properties such as lengths and angles because they have already been captured by geometry.



  - Line segments with different lengths are topologically same shapes, if we bend it to obtain a line segment to obtain the third or fourth shape, it will stay the same topological shape, as long as we have not tear it into parts or glue its ends.

Definition 1.1: *Two objects are* **topologically identical** *if there is a continuous deformation (such as bending and stretching) from one to the other.*

Overview



Think about these figures, are they topologically equivalent?!!!

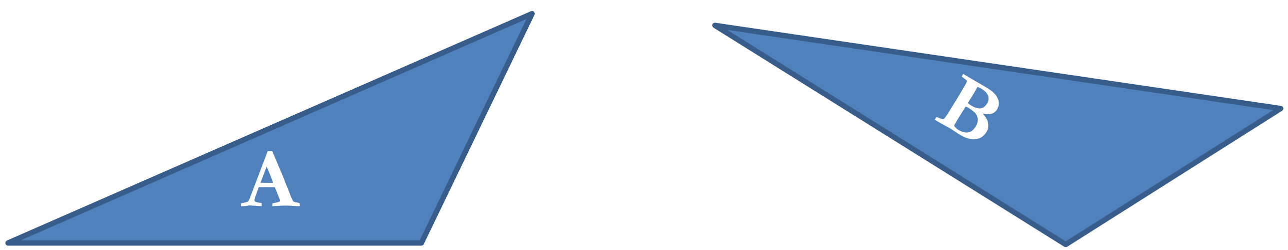
Overview

Definition 1.2: *An* **equivalence relation**, $\sim$, *on a set of objects* A *is a relation on this set such that:*

(1) *For each* $x \in A$ *we have* $x \sim x$ *(reflexitivity).*

(2) *For* $x, y \in A$, *if* $x \sim y$ *then* $y \sim x$ *(symmetry).*

(3) *For* $x, y, z \in A$, *if* $x \sim y$ *and* $y \sim z$, *then* $x \sim z$ *(transitivity).*



- A **geometric equivalence relation** has nothing to do with the positioning of the figures, i.e. two congruent triangles can have vertices with different coordinates. Hence congruence only conveys information related to the geometric properties possessed by the figures.

Overview

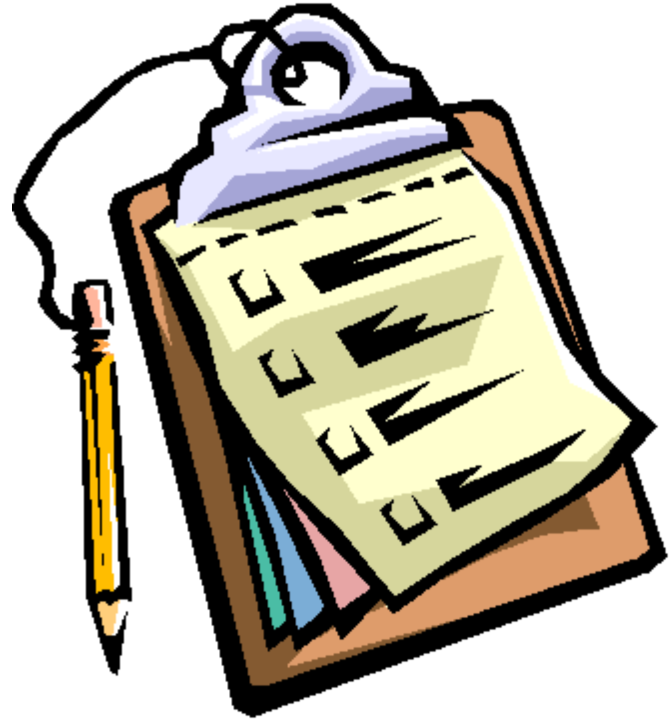
- Figures as triangles and circles are considered as geometric objects, while topological spaces are the objects of topology.

Definition 1.3: *Let* $\mathcal{X}$ *be a set and* $\mathcal{T}$ *a collection of subset of* $\mathcal{X}$. *The collection* $\mathcal{T}$ *is called a* **topology** *on* $\mathcal{X}$ *if:*

(1) The empty set $\emptyset$ *and* $\mathcal{X}$ *are in* $\mathcal{T}$.
(2) The union of an arbitrary collection of members (sets) of $\mathcal{T}$ *is in* $\mathcal{T}$.
(3) The intersection of any finite collection of sets in $\mathcal{T}$ *is also in* $\mathcal{T}$.

The pair $(\mathcal{X}, \mathcal{T})$ *is called a* **topological space**.

Agenda



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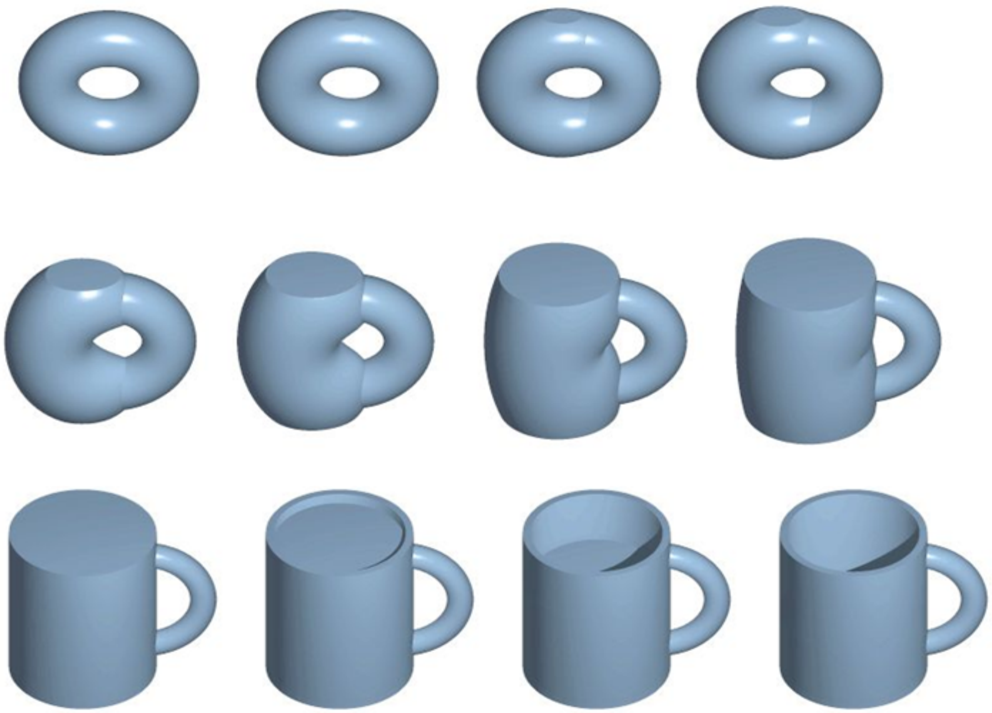
Functions

- There is a common joke that *topologists are mathematicians who cannot tell their donut from their coffee cups*, this is simply because there is a continuous function which deform a donut to a coffee cup.

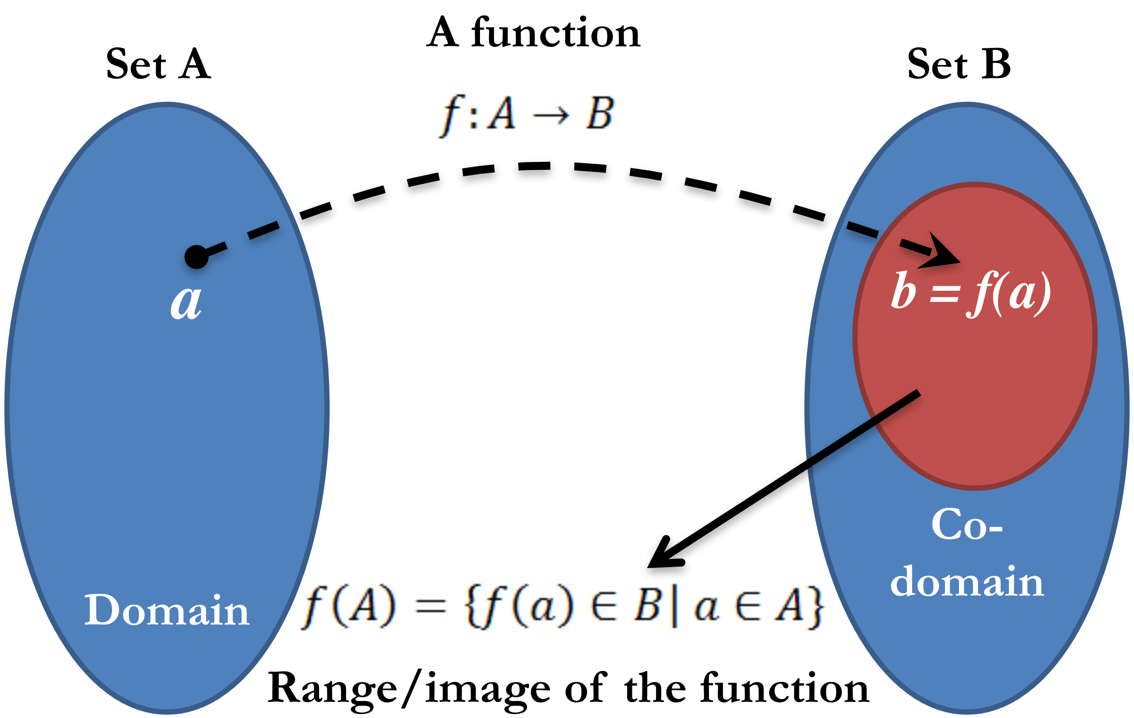


A donut being deformed to form a coffee cup, images taken from Wikipedia

Functions

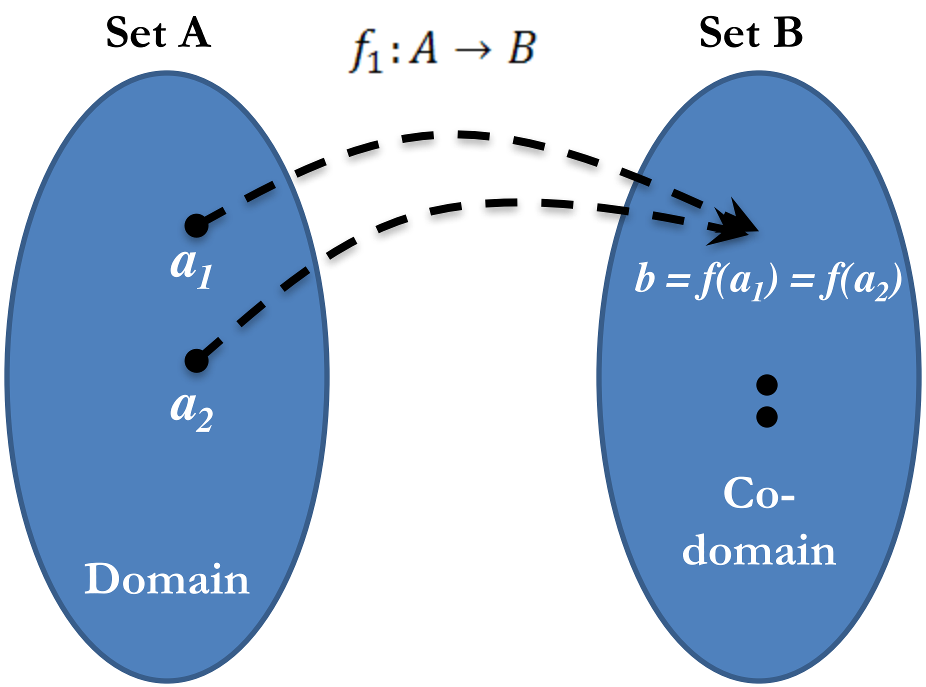
- Thus functions are the core of topology, it can be defined as follows.

Definition 1.4: *A* **function** f *is a well-defined rule assigning to each element of a set* A *a unique element in the set* B*, it is denoted by* $f: A \rightarrow B$*. The set* A *is the* **domain** *of the function* f *and the receiving set* B *is its* **codomain**. *The* **image** *or* **range** *of the function can be defined as* $f(A) = \{f(a) \in B \mid a \in A\}$ *which is a subset of* B.

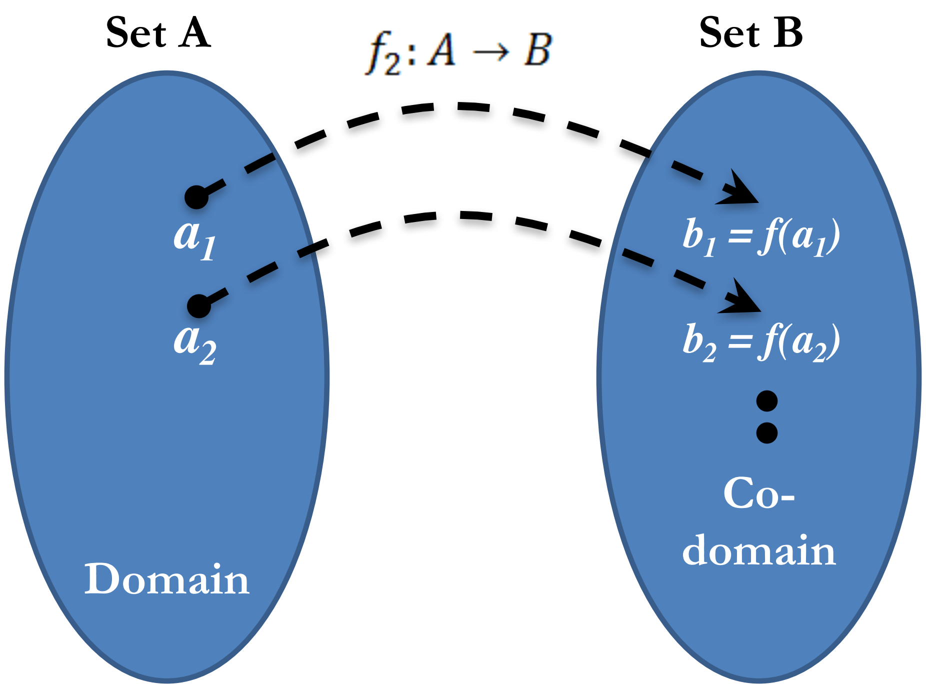


Functions

Definition 1.5: *A function* $f: A \to B$ *is* **one-to-one** *(injection - monomorphism) if whenever* $f(a_1) = f(a_2)$, *then we have* $a_1 = a_2$.



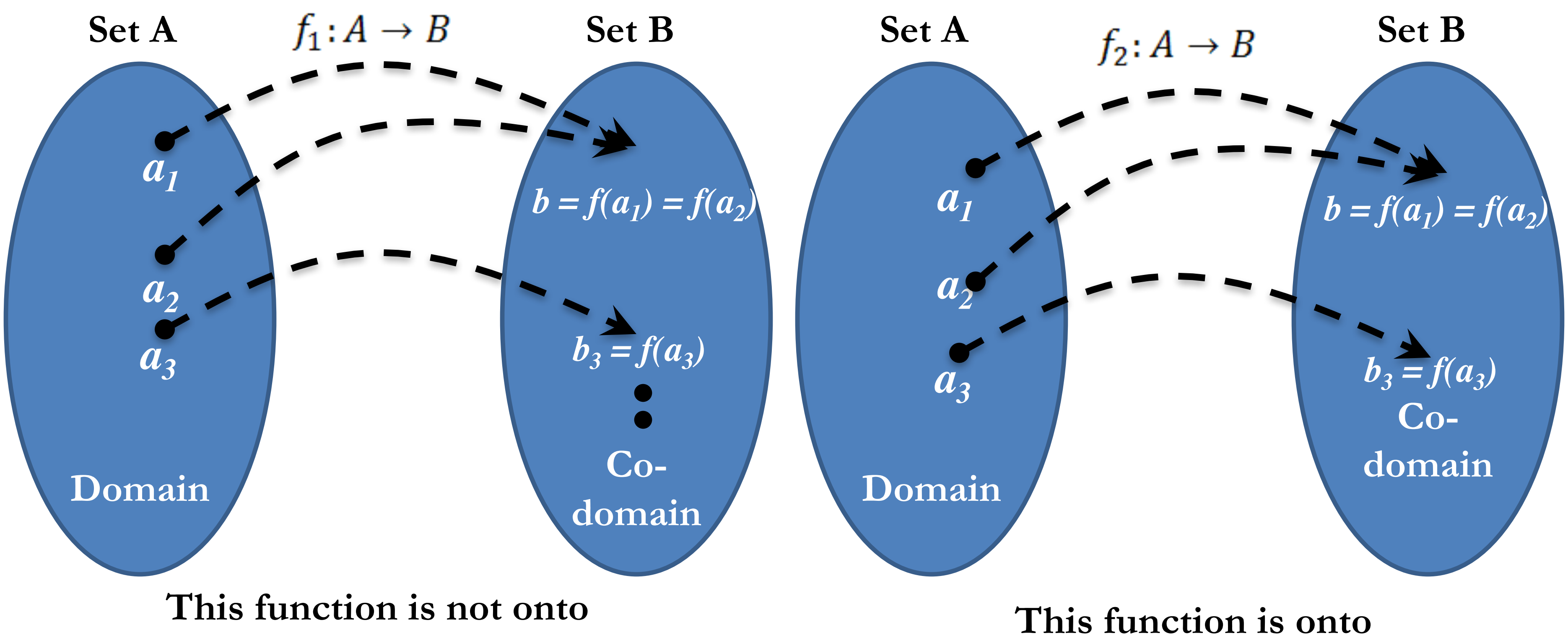
This function is not one-to-one



This function is one-to-one

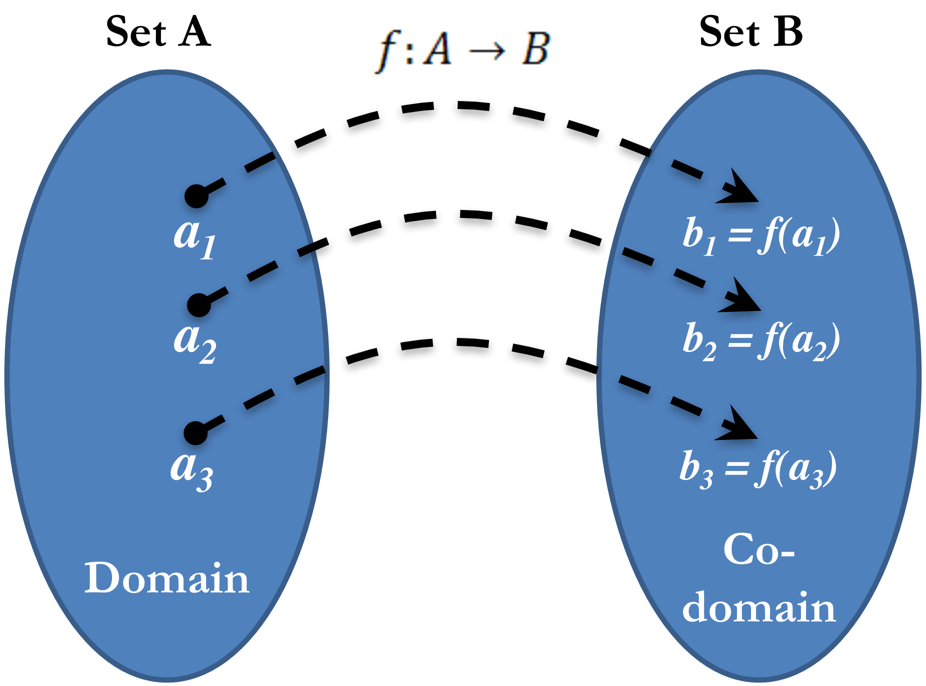
Functions

Definition 1.6: *A function* $f: A \to B$ *is* **onto** *(surjection - epimorphism) if for any* $b \in B$, *there is an* $a \in A$ *with* $f(a) = b$



Functions

Definition 1.7: *A function* $f: A \to B$ *is* **onto correspondence** *(bijection) if* f *is both one-to-one and onto.*



This function is a bijection

Functions

Definition 1.8: *If* $f: A \to B$ *is a onto correspondence (bijection), then* f *has an inverse* $f^{-1}: B \to A$*, which is determined by the fact that if* $b \in B$*, then there is an element* $a \in A$ *such that* $f(a) = b$*. Furthermore* a *is uniquely determined by* b *because* $f(a) = f(a') = b$ *implies that* $a = a'$*, hence we can define* $f^{-1}(b) = a$*.*

- There is another way to define the inverse of a function which mainly depends on the so-called identity function.

Definition 1.9: *Let* f *and* g *be two functions such that* $f: A \to B$ *and* $: B \to C$ *, then the* **composition** $g \circ f: A \to C$ *is defined for any* $x \in A$ *by* $g \circ f(x) = g(f(x)) \in C$*.*

Definition 1.10: *A function* $f: A \to A$ *defined by* $f(x) = x$ *is called the* **identity function** *and is denoted by* id_A*.*

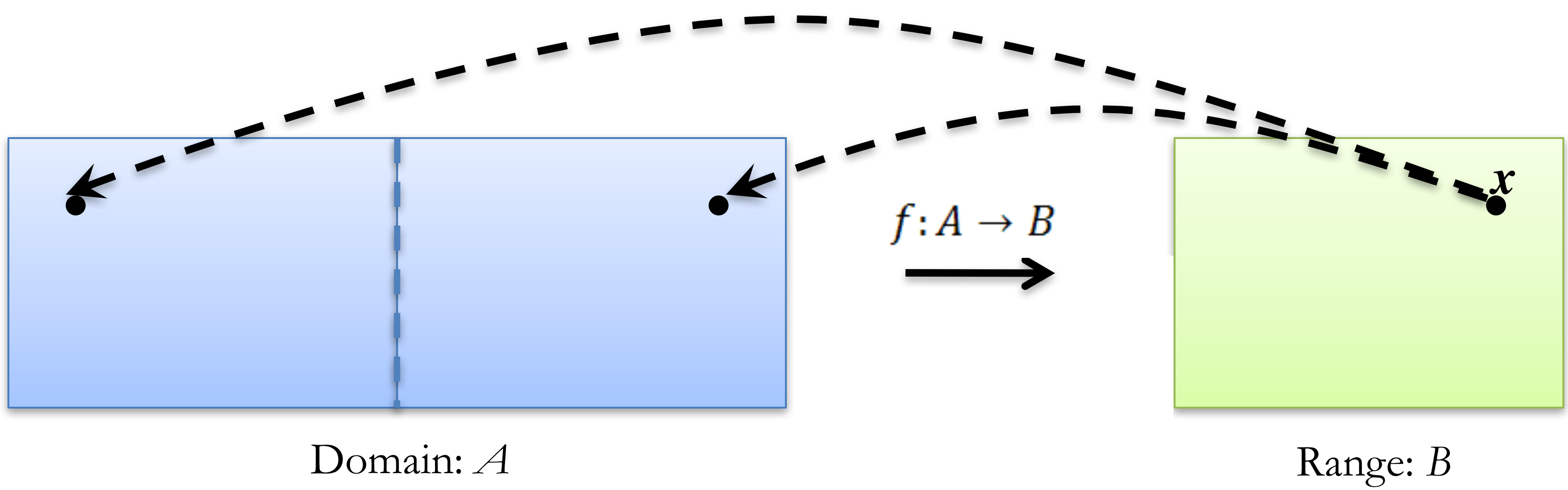
Definition 1.11: *A function* $f: A \to B$ *is* **invertible** *if there is a function* $g: B \to A$ *such that*

$$id_B = f \circ g: B \to B \text{ and } id_A = g \circ f: A \to A$$

The function g *is called the* **inverse** *of* f *and is usually denoted by* f^{-1}*.*

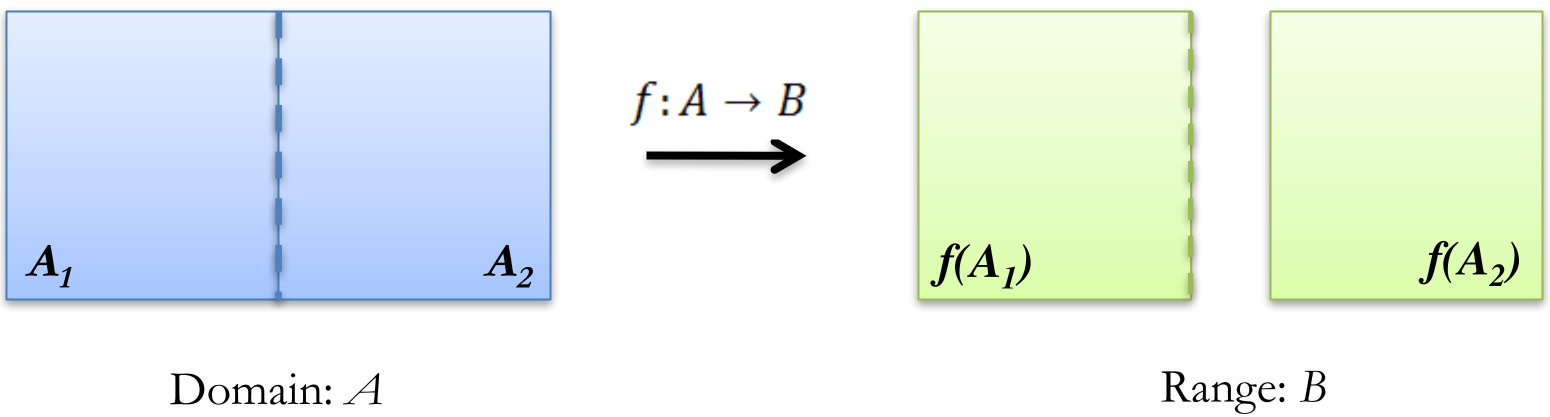
Functions

Example: Consider the function f that folds the rectangle A along the dotted line, f is not invertible since if an inverse f^{-1} does exist, it would take the square $B = f(A)$ and give A back, but to do so, the point x would have to go to both of the points in$A$, thus f^{-1} is not a function.



Functions

However, the function f that takes a rectangle and cut it in half along the dotted line is onto and one-to-one, thus invertible. The inverse would take the two squares and glue them back together; however this function is not continuous.



Functions

- Now it is time to define the continuity of a function, we will first consider continuity of a single-variable function.

Definition 1.11: *Let* $f: \mathbb{R} \to \mathbb{R}$ *be a single-variable function, where* $\mathbb{R}$ *is the set of real numbers, this function is continuous at a point* $x_o \in \mathbb{R}$ *if for every* $\varepsilon > 0$, *there is* $\delta > 0$, *such that whenever* $|x - x_o| < \delta$, *we have* $|f(x) - f(x_o)| < \varepsilon$

- In order to generalize this definition for other types of function with arbitrary domain other than the real line, we need first to define a distance measure

Definition 1.12: *A* **metric space** *is a set* $\mathcal{X}$ *together with a distance function* $\mathrm{d}: \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ *satisfying the following:*

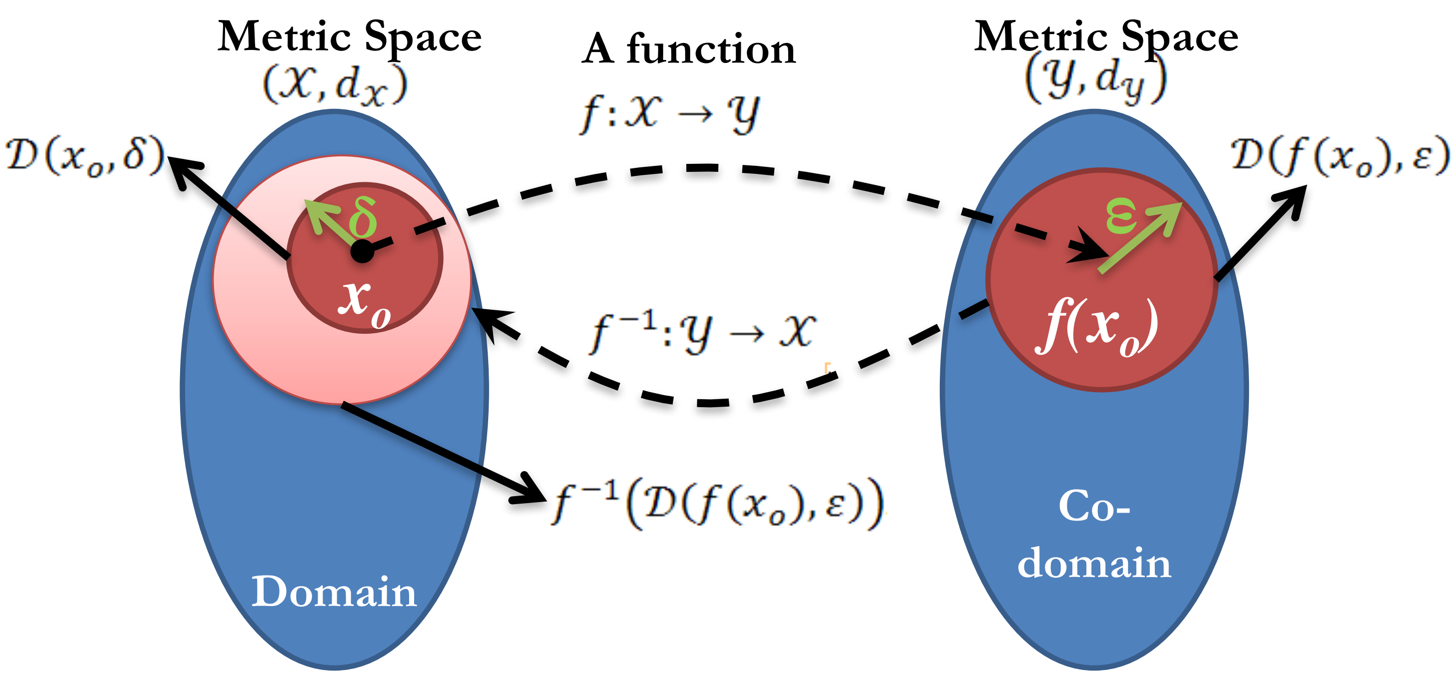
(1) $d(x, y) \geq 0$ *for all* $x, y \in \mathcal{X}$.
(2) $d(x, y) = 0$ *if and only if* $x = y$
(3) $d(x, y) = d(y, x)$ *for all* $x, y \in \mathcal{X}$.
(4) The triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$ *for all* $x, y, z \in \mathcal{X}$.

Functions

Definition 1.13: *The* **open disc** *or* **open ball** *centered at* $\mathbf{x}$ *in a metric space* $(\mathcal{X}, d)$ *with radius* ε *is given by* $\mathcal{D}^n(\mathbf{x}, \varepsilon) = \{\mathbf{y} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{y}) < \varepsilon\}$ *that is the points in* $\mathcal{X}$ *within* ε *in distance from* $\mathbf{x}$.

- Now we can re-write the definition of continuity as follows.

Definition 1.14: *Suppose that* $(\mathcal{X}, d_{\mathcal{X}})$ *and* $(\mathcal{Y}, d_{\mathcal{Y}})$ *are two metric spaces and* $f : \mathcal{X} \rightarrow \mathcal{Y}$ *is a function. Then* f *is* **continuous** *at* $x_o \in \mathcal{X}$ *if, for any* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that* $\mathcal{D}(x_o, \delta) \subset f^{-1}(\mathcal{D}(f(x_o), \varepsilon))$. *The function* f *is continuous if it is continuous at* x_o *for all* $x_o \in \mathcal{X}$.



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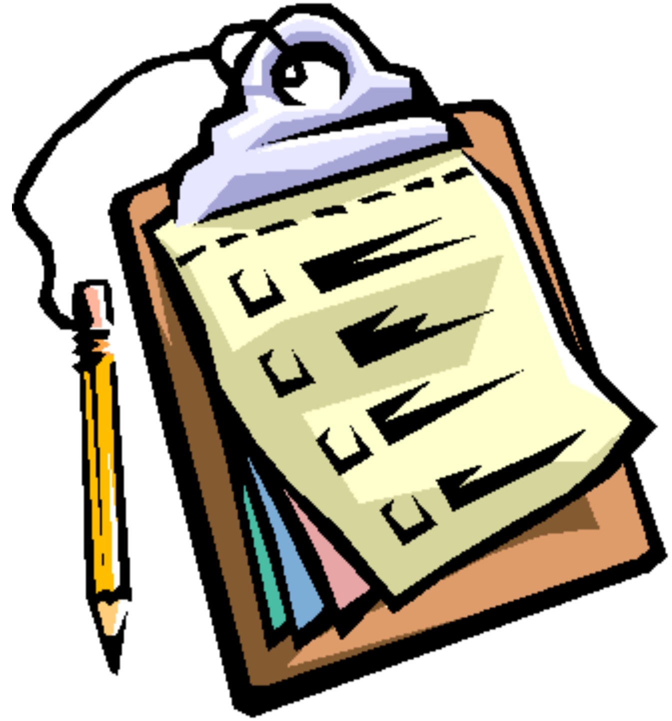

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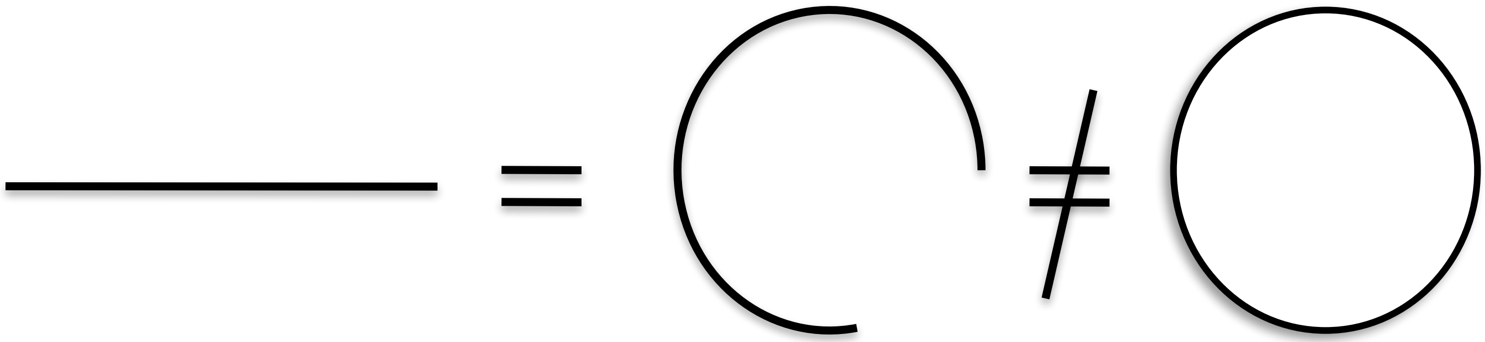
Topological equivalence

Definition 1.15: *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be topological spaces. Then* $\mathcal{X}$ *is* **topologically equivalent** *or* **homeomorphic** *to* $\mathcal{Y}$ *if there is a continuous invertible function* $f: \mathcal{X} \to \mathcal{Y}$ *with continuous inverse* $f^{-1}: \mathcal{Y} \to \mathcal{X}$. *Such a function* f *is called a* **homeomorphism.**



- A circular arc is topologically equivalent to a line segment but not topologically equivalent to a circle. You can also view this figure in the other way, if we cut a circle we will form a circular arc, but his operation is not continuous.

Theorem 1.1: *Topological equivalence is an equivalence relation*

- This theorem implies that one can use topological equivalence like equality, topologically equivalent spaces have the same topological shapes.

Topological equivalence

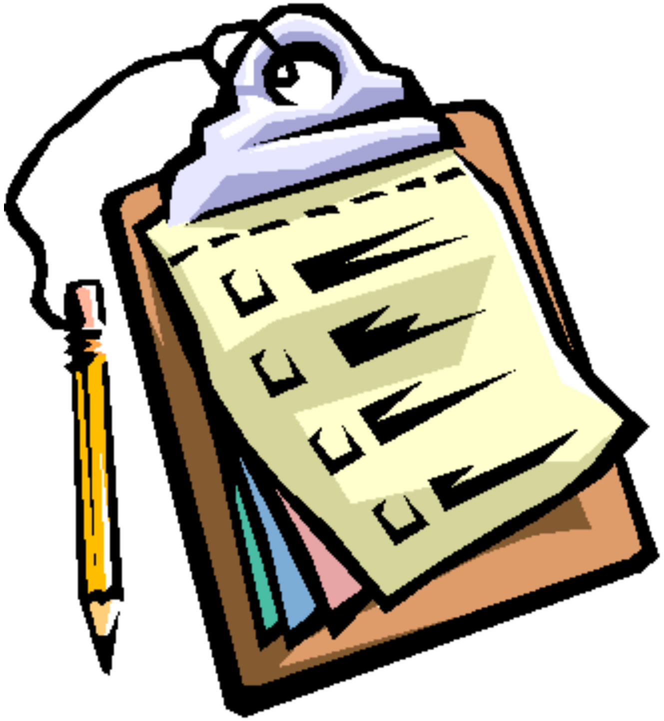
Definition 1.16: *Let* x *be a point on the sphere* $\mathbb{S}^2$*, then* $\mathbb{S}^2 - \{x\}$ *is homeomorphic to* $\mathbb{R}^2$ *by* **stereographic projection,** *this is defined by placing the sphere on the plane so that they are tangent and the puncture is at the North pole, for each* $y \in \mathbb{S}^2$*, note that the ray from* x *through* y *intersects the plane* $\mathbb{R}^2$ *at a unique point. Define* $s(y)$ *to be the point where this ray intersects the plane.*

Stereographic projection can be thought of as stretching the punctured sphere and laying it out flat. Thus the punctured sphere $\mathbb{S}^2 - \{x\}$ and the plane $\mathbb{R}^2$ are topologically equivalent, however one space is bounded and the other is unbounded.

Definition 1.17: *Property* $\mathbb{P}$ *is a topological property if whenever set* A *has property* $\mathbb{P}$ *and set* B *is topologically equivalent to* A*, then* B *also has property* $\mathbb{P}$.

Thus boundedness is not a topological property.

Agenda



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Now, it is time to specify the category of the object to be investigated. In topology, the most general object to deal with is a set of points on which functions can be defined.

Open and Closed Sets in $\mathbf{R}^n$

- We will consider point sets which are subsets of the real Euclidean *n*-space which is defined as follows.

Definition 1.18: *Real Euclidean n-space is given by* $\mathbb{R}^n = \{\mathbf{x} = (x_1, x_2, \dots, x_n) |\ x_i \in \mathbb{R}\}$ *where* $\mathbf{x}$ *denotes a point with n-coordinates.*

Example: $\mathbb{R}^1 = \mathbb{R}$ is the real line while $\mathbb{R}^2 = \{\mathbf{x} = (x, y)\}$ is the standard Euclidean plane and $\mathbb{R}^3 = \{\mathbf{x} = (x, y, z)\}$ is the three dimensional space. In general we use x to denote a real number while $\mathbf{x}$ to denote a point in *n*-dimensional space where $n > 1$.

- We can define a distance measure for Euclidean space using the Euclidean metric which is given by the following definition.

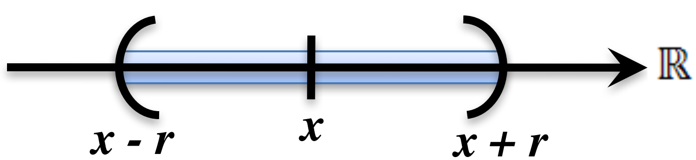
Definition 1.19: *Given two points* $\mathbf{x} = (x_1, x_2, \dots, x_n)$ *and* $\mathbf{y} = (y_1, y_2, \dots, y_n)$ *in* $\mathbb{R}^n$, *the* **Euclidean distance/metric** *between* $\mathbf{x}$ *and* $\mathbf{y}$ *is given by*

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

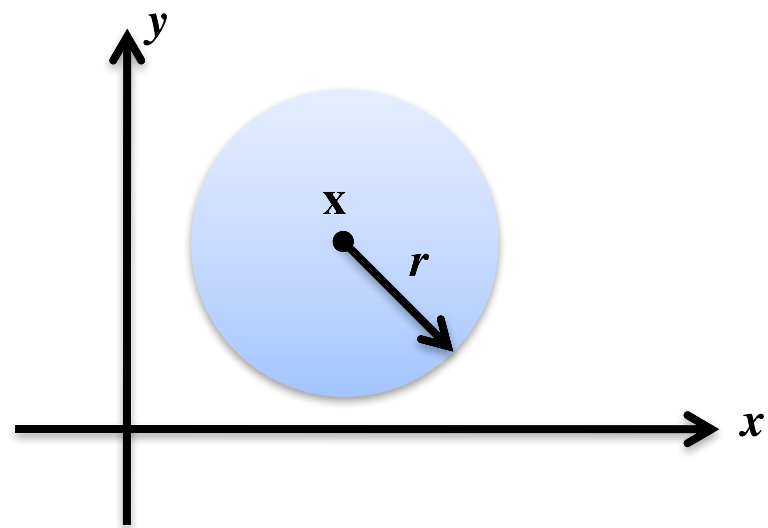
Open and Closed Sets in $\mathbf{R}^{n}$

- Hence, a disc or ball centered at with radius *r* can be defined as:

$$\mathcal{D}^n(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{y}\| < r\}.$$



$$\mathcal{D}^1(x, r) = \{y \in \mathbb{R} : \|x - y\| < r\}$$



$$\mathcal{D}^2(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^2 : \|\mathbf{x} - \mathbf{y}\| < r\}$$

Definition 1.20: $\mathcal{D}^n(\mathbf{x}, r)$ *is called (disc)* **neighborhood** *of* $\mathbf{x}$ *in* $\mathbb{R}^n$ *which is an open disc containing* $\mathbf{x}$.

Open and Closed Sets in R^n

- If we have a set of points in the Euclidean *n*-space, a point can be related to this set in terms of its neighborhood in one of the following ways.

Definition 1.21: *Let* A *be a set of points with* $A \subseteq \mathbb{R}^n$ *and* $\mathbb{R}^n - A$ *be all points not in* A*, i.e. the complement of* A*:*

(1) *A point* $\mathbf{x} \in \mathbb{R}^n$ *is an* **interior** *point of* A *if there is a neighborhood* $\mathcal{N}$ *of* $\mathbf{x}$ *such that* $\mathbf{x} \in \mathcal{N} \subseteq A$*, i.e. the disc is totally enclosed in* A*.*

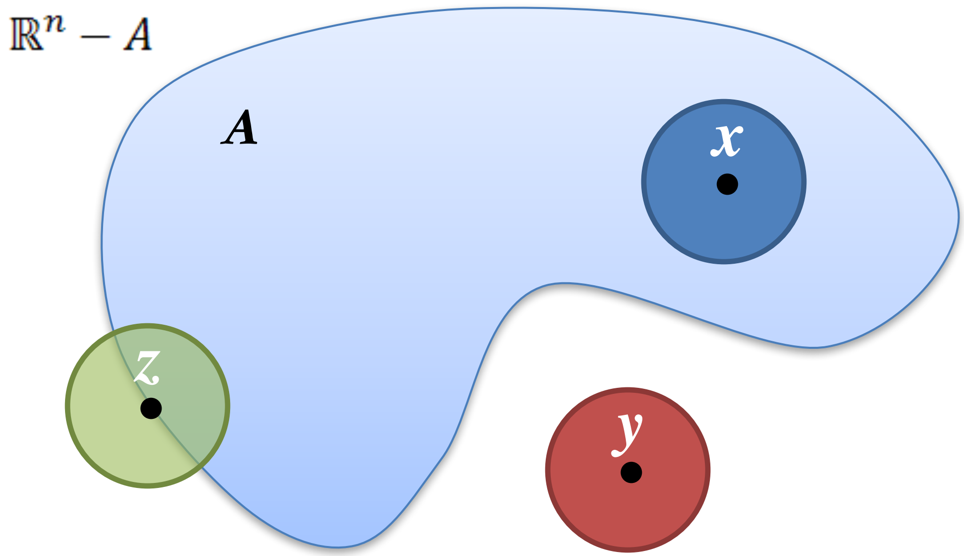
(2) *A point* $\mathbf{x} \in \mathbb{R}^n$ *is an* **exterior** *point of* A *if there is a neighborhood* $\mathcal{N}$ *of* $\mathbf{x}$ *such that* $\mathbf{x} \in \mathcal{N} \subseteq (\mathbb{R}^n - A)$*, i.e. the disc is totally outside* A*. Another way of saying this is* $\mathcal{N} \cap A = \phi$*.*

(3) *A point* $\mathbf{x} \in \mathbb{R}^n$ *is a* **limit/accumulation** *point of* A *if every neighborhood* $\mathcal{N}$ *of* $\mathbf{x}$ *contains at least one point of* A*, thus* $\mathcal{N} \cap A \neq \phi$*. Hence every point of* A *is a limit point of* A*.*

(4) *A point* $\mathbf{x} \in \mathbb{R}^n$ *is an* **isolated** *point of* A *if* $\mathbf{x} \in A$ *and it has a neighborhood* $\mathcal{N}$ *satisfying* $\mathcal{N} \cap A = \{\mathbf{x}\}$*. Hence a point* $\mathbf{x}$ *is not isolated if every neighborhood of* $\mathbf{x}$ *contains at least one point in* A *other than itself.*

(5) *A point* $\mathbf{x} \in \mathbb{R}^n$ *is a* **boundary/frontier** *point of* A *if every neighborhood* $\mathcal{N}$ *of* $\mathbf{x}$ *intersects both* A *and* $\mathbb{R}^n - A$*, i.e. contains points in and outside* A*. Thus boundary points are also limit points.*

Open and Closed Sets in R^n



- Point x is an interior point of A since it has a neighborhood totally enclosed in A, point y is an exterior point of A since it have a neighborhood which lies outside A, while z is a boundary point of A because it has a neighborhood which contains points in and outside A.

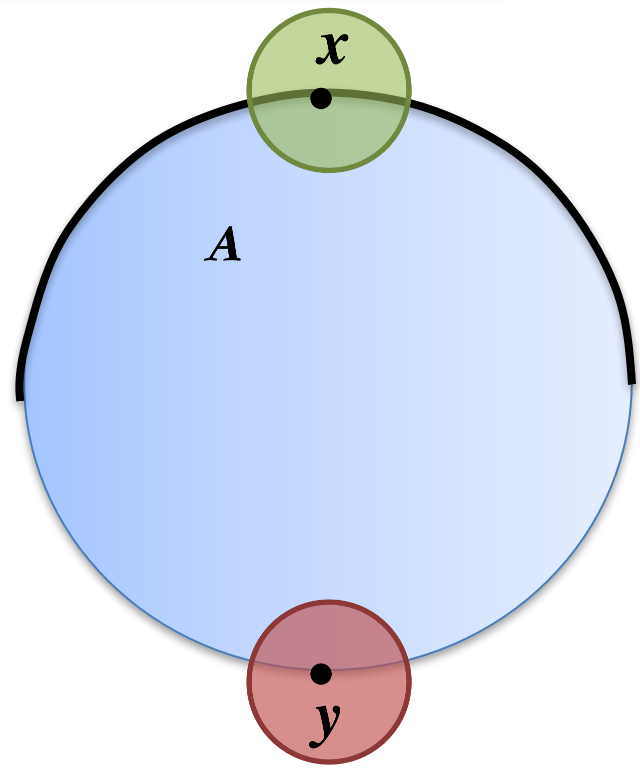
Open and Closed Sets in R^n

- Now, we are ready to define open and closes sets.

Definition 1.22: *A set* A *is said to be* **open** *if every point* $\mathbf{x} \in A$ *is an interior point.*

Definition 1.23: *A set* A *is said to be* **closed** *if every point* $\mathbf{x} \notin A$ *is an exterior point.*

- Let A be a disc including the upper semicircle but not the lower one, x is a point in A which is not an interior point of A (it is a boundary point) so A cannot be an open set, on the other hand point y is not in A but it is also not an exterior point to A, hence A is not a closed set, thus being not an open set does not mean to be a closed one.



Open and Closed Sets in $\mathbf{R}^n$

Definition 1.24: *A set $A \subseteq \mathbb{R}^n$ is said to be* **bounded** *if $A \subseteq \mathcal{D}^n(\mathbf{0}, r)$ for some r. Thus the set A can be enclosed in some sufficiently large disc centered at its origin, i.e. A does not go on forever.*

Definition 1.25: *The* **interior** *of a set A, denoted by $Int(A)$, is the set of all interior points of A. Thus $Int(A) \subseteq A$.*

If A consist of a single point $\mathbf{x}$**,** then $Int(A) = \phi \subseteq A = \{\mathbf{x}\}$

Definition 1.26: *The* **boundary** *of a set A, denoted by $bdy(A)$, is the set of all boundary points of A. It is also known as the frontier of the set A, denoted by $Fr(A)$.*

Definition 1.27: *The* **closure** *of a set A, denoted by $Cl(A)$, is the set containing the points which lies in A and the points on the boundary of A, i.e. $A \cup bdy(A)$. Thus for any set A, we have $A \subseteq Cl(A)$.*

Theorem 1.2: *For any set $A \subseteq \mathbb{R}^n$, its closure $Cl(A)$ is a closed set.*

Definition 1.28: *A* **sequence** *in A is an infinite ordered set of points $\{x_i\}_{i=1}^{\infty} = \{x_1, x_2, x_3, \dots\}$ where $x_i \in A \subseteq \mathbb{R}^n$.*

Definition 1.29: *A point x is a* **limit point** *of a sequence $\{x_i\}_{i=1}^{\infty}$ if every neighborhood of x contains an infinite number of x_i's.*

Open and Closed Sets in $\mathbf{R}^n$

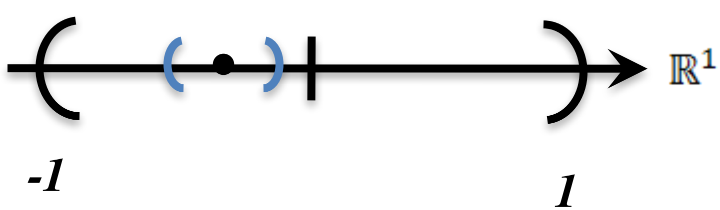
There is a difference between a sequence and a set, consider $\{(-1)^k\}_{k=1}^{\infty}$ in $\mathbb{R}$, as a set this consists of only two elements $\{-1,1\} = B$, but as a sequence this is the infinite list $\{-1,1,-1,1,\dots\}$, this sequence has two limit points -1 and 1, both of which are also the limit points of the set B.

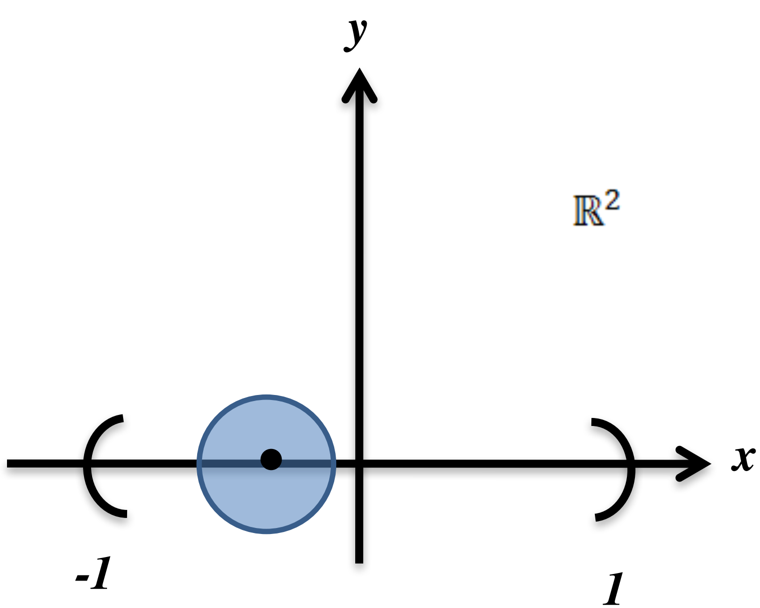
Theorem 1.3: *If* x *is a limit point of a set* $A \subseteq \mathbb{R}^n$, *then there is a sequence of points* $\{x_i\}_{i=1}^{\infty}$, *where* $x_i \in A$, *such that* x *is a limit point of the sequence* $\{x_i\}_{i=1}^{\infty}$.

Theorem 1.4: *If* $\{x_i\}_{i=1}^{\infty}$ *is a sequence with each* $x_i \in A \subseteq \mathbb{R}^n$ *and* x *is a limit point of the sequence* $\{x_i\}_{i=1}^{\infty}$, *then* x *is a limit point of the set* A.

Relative Neighborhoods

Consider the open interval (-1,1) in $\mathbb{R}^1$ which can be defined as $\mathcal{D}^1(\mathbf{0},\mathbf{1})$, this interval is open when considered as a subset of $\mathbb{R}^1$ however when considered as a subset of $\mathbb{R}^2$, i.e. a plane, the interval is not open any more since any disc about a point in (-1,1) will overlap with the upper and lower half planes. Hence the neighborhood should be defined within a context, see the following definition.



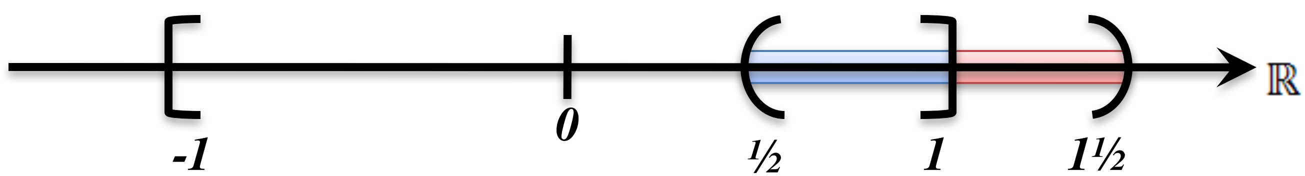


Definition 1.30: *Let* $A \subseteq \mathbb{R}^n$. *A neighborhood of a point* $\mathbf{x} \in A$ *relative to* A *is a set of the form* $\mathcal{D}^n(\mathbf{x}, r) \cap A$.

Relative Neighborhoods

Examples

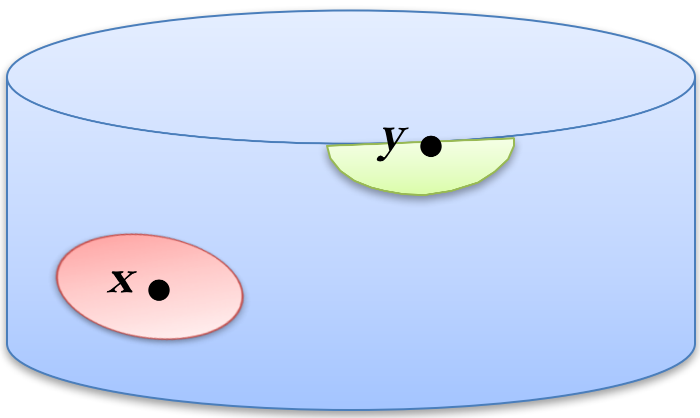
(1) consider the interval $[-1,1]$, let $x = 1 \in [-1,1]$, if we consider its neighborhood as $\mathcal{D}^1\left(1,\frac{1}{2}\right)$, i.e. open interval centered at 1 with radius ½ , a part of the neighborhood will lie outside the interval $[-1,1]$, hence the point $x = 1$ has the set $(\frac{1}{2},1]$ as a neighborhood relative to $[-1,1]$, since $\mathcal{D}^1\left(1,\frac{1}{2}\right) \cap [-1,1] = (\frac{1}{2},1]$, moreover the set $(\frac{1}{2},1]$ is considered open in $[-1,1]$ since every point in $(\frac{1}{2},1]$ has a neighborhood (relative to $[-1,1]$) totally enclosed in $(\frac{1}{2},1]$, but $(\frac{1}{2},1]$ is not an open set when viewed as a subset of $\mathbb{R}$.



Relative Neighborhoods

Examples

(2) Consider a cylinder viewed as a subset of $\mathbb{R}^3$. The neighborhood of the point x with respect to $\mathbb{R}^3$ is supposed to be an open ball/sphere, however when viewed to be a point on the surface of the cylinder, the neighborhood of x relative to the cylinder is a warped disk which is the intersection of the cylinder surface with an open ball in $\mathbb{R}^3$ centered at x. In the same manner, the neighborhood of y is a warped half disk.



Relative Neighborhoods

- Now we can consider reformulating the definitions of interior, exterior and limit points, since their definitions only refer to neighborhoods not specifically to discs, these definitions remain valid for relative neighborhoods.

Definition 1.31: *Let $B \subseteq A$;*

(1) A point $x \in A$ is an **interior point of B relative to A** *if there is a neighborhood $\mathcal{N}$ of x relative to A such that $x \in \mathcal{N} \subseteq B$, i.e. the neighborhood is totally enclosed in B.*
(2) A point $x \in A$ is an **exterior point of B relative to A** *if there is a neighborhood $\mathcal{N}$ of x relative to A such that $x \in \mathcal{N} \subseteq A - B$, i.e. the neighborhood is totally outside B.*
(3) A point $x \in A$ is a **limit point of B relative to A** *if there is a neighborhood $\mathcal{N}$ of x relative to A such that $\mathcal{N} \cap B \neq \emptyset$.*

- Hence the definitions of open sets can be reformulated as follows;

Definition 1.32: *Let $B \subseteq A$, the set B is* **open relative to** *A if every point $x \in B$ is an interior point to B relative to A, i.e. if every point $x \in B$ has a relative neighborhood $\mathcal{N}$ with $x \in \mathcal{N} \subseteq B$.*

Relative Neighborhoods

- Another way of defining relatively open and closed sets is the following theorem.

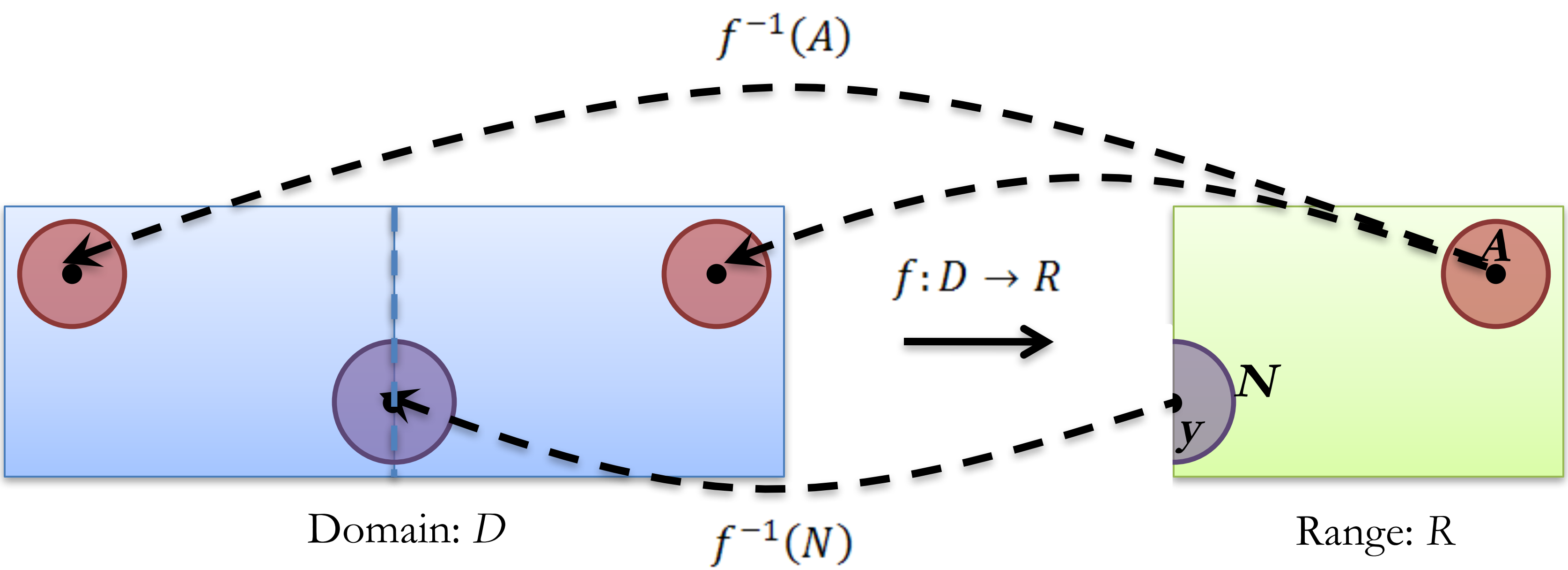
Theorem 1.5: *Let* $B \subseteq A \subseteq \mathbb{R}^n$,

(1) $\boldsymbol{B}$ **is open relative to** $\boldsymbol{A}$ *if and only if* $B = A \cap O$ *for some set* O *which is open in* $\mathbb{R}^n$.
(2) $\boldsymbol{B}$ **is closed relative to** $\boldsymbol{A}$ *if and only if* $B = A \cap C$ *for some set* C *which is closed in* $\mathbb{R}^n$.

Continuity

- We will reformulate the definition of function continuity in terms of open and closed sets.

Definition 1.33: *Let* $D \subseteq \mathbb{R}^n$ *and* $R \subseteq \mathbb{R}^m$. *A function* $f : D \rightarrow R$ *n is* **continuous** *if whenever* A *is an open set in* R, *then* $f^{-1}(A)$ *is an open set in* D, *where* f^{-1} *denotes the set-theoretic inverse of* f, *such that* f^{-1} *is defined for any subset* A *in the range of* f *by* $f^{-1}(A) = \{\mathbf{x} \in D : f(\mathbf{x}) \in A\}$.



Compact Sets

Definition 1.34: *A set A is (sequentially)* **compact** *if every infinite sequence of points in A has a limit point in A, that is if $\{x_i\}_{i=1}^{\infty}$ is a sequence and $x_i \in A$ for each i, then there is a point $x \in A$ such that x is a limit point of $\{x_i\}_{i=1}^{\infty}$.*

Examples:

(1) The real line $\mathbb{R}$ is not compact, since the sequence $\{1,2,3,\dots\} = \{n\}_{n=1}^{\infty}$ consists of points in $\mathbb{R}$ but has no limit points in $\mathbb{R}$.

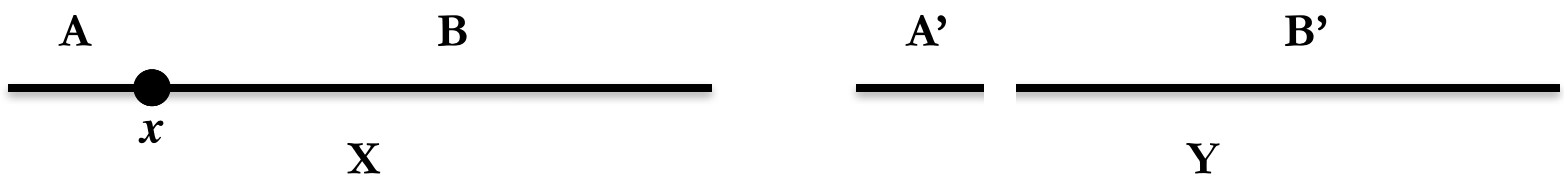
(2) The interval $(0,1)$, the number 0 is a limit point for the sequence $\left\{\frac{1}{2},\frac{1}{3},\frac{1}{4},,\dots\right\} = \{\frac{1}{n}\}_{n=2}^{\infty} \subseteq (0,1)$, but $0 \notin (0,1)$, hence the open interval $(0,1)$ is not compact.

Theorem 1.7: Heine-Borel Theorem – *A set $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

- The open interval (-1,1) is bounded but not finite (not compact), imagine walking along this interval with an exponentially decaying step, as you move towards 1, your step gets smaller, hence one will never reach 1, in this sense (-1,1) is endless hence infinite. On the other hand, the closed interval [-1,1] is compact and does not go forever since it has ends. Thus topological properties should not be based on distance along as in boundedness since in topology distances mean very little.

Connected Sets

- Another fundamental notion in topology is the number of pieces/components an object has, if an object contains only one piece, it is considered connected; this is true if all its parts are stuck to each other.



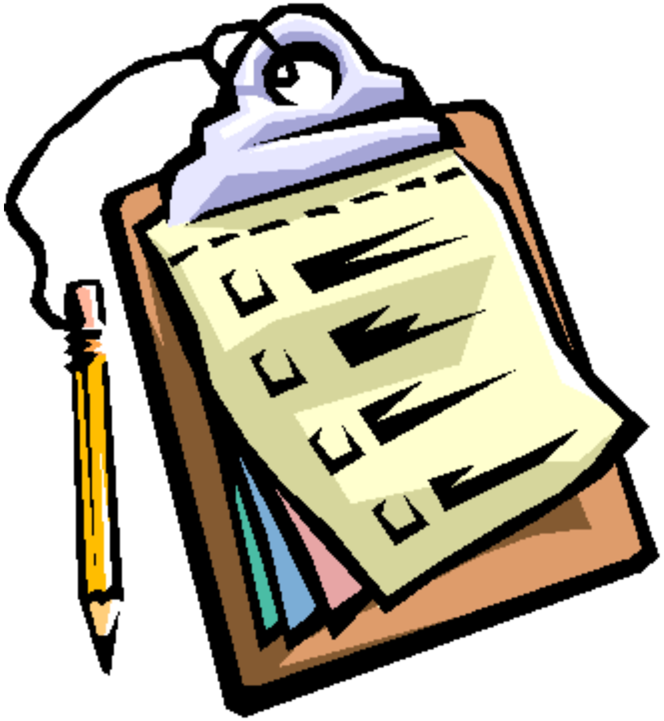
  - X and Y are divided into two parts, where B looks exactly like B’ and A looks like A’ with the point x added on, x can be viewed as gluing A and B together to make X connected, x is absent from Y, hence there is a gap in Y, i.e. Y has two pieces. Note that x is in the set A and is a limit point of both A and B, so that x cannot be separated from B so X is connected while Y is not connected.

Definition 1.35: *A set S is* **connected** *if whenever S is divided into two non-empty sets such that* $S = A \cup B, A \neq \emptyset, B \neq \emptyset$ *and* $A \cap B = \emptyset$*, then either* A *or* B *contains a limit point of the other.*

  - **Example:** the interval [0,1] is connected.

Theorem 1.8: *If* $f: A \to B$ *is a continuous function from a connected set* A *onto the set* B *then* B *is connected.*

Agenda



- Overview
- Functions
- Topological equivalence
- Point-set topology in R^n
- **Point-set topology – more abstract level**
- Surfaces

So far, all examples and theorems dealt with sets in R^n and used the structure of the standard Euclidean structure, in particular, the definitions of neighborhoods which makes use of Euclidean distance, however studying point-set topology on a more abstract level requires different definition of neighborhood.

Open Sets and Neighborhoods

- Recall:

Definition 1.12: *A* **metric space** *is a set* $\mathcal{X}$ *together with a distance function* $\mathbf{d}: \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ *satisfying the following:*

(1) $d(x, y) \geq 0$ *for all* $x, y \in \mathcal{X}$.
(2) $d(x, y) = 0$ *if and only if* $x = y$
(3) $d(x, y) = d(y, x)$ *for all* $x, y \in \mathcal{X}$.
(4) The triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$ *for all* $x, y, z \in \mathcal{X}$.

- Definition 1.12 defines metric space in terms of metric/distance measure *d*, the metric topology defined on a metric space can be defined as follows.

Definition 1.36: *Let* $\mathcal{X}$ *be a metric space, the metric topology on* $\mathcal{X}$ *is defined by using sets* $\mathcal{N}$ *as neighborhoods of a point* $x \in \mathcal{X}$ *where* $\mathcal{N} = \mathcal{D}_{\mathcal{X}}(\mathbf{x}, r) = \{y \in \mathcal{X}: d(x, y) < r\}$, *with r as a real number greater than zero.*

  - Now any distance function defines a collection of neighborhoods which in turn defines open, closed sets and continuous functions.
  - Thus we can redefine everything in terms of more general open sets and neighborhoods starting with the definition of a topological space.

Open Sets and Neighborhoods

Definition 1.37: *A* **topological space** *is a set* $\mathcal{X}$ *with a collection* $\mathcal{B}$ *of subsets* $\mathcal{N} \subseteq \mathcal{X}$ *called neighborhood such that:*

(1) *Every point is in some neighborhood, i.e. for every* $x \in \mathcal{X}$, *there exist a neighborhood* $\mathcal{N} \in \mathcal{B}$ *such that* $x \in \mathcal{N}$.

(2) *The intersection of any two neighborhoods of a point contains a neighborhood of the point, i.e. for every* $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{B}$ *with* $x \in \mathcal{N}_1 \cap \mathcal{N}_2$, *there exist* $\mathcal{N}_3 \in \mathcal{B}$ *such that* $x \in \mathcal{N}_3 \subseteq \mathcal{N}_1 \cap \mathcal{N}_2$.

The set $\mathcal{B}$ *of all neighborhoods is called a* **basis** *for the topology on* $\mathcal{X}$.

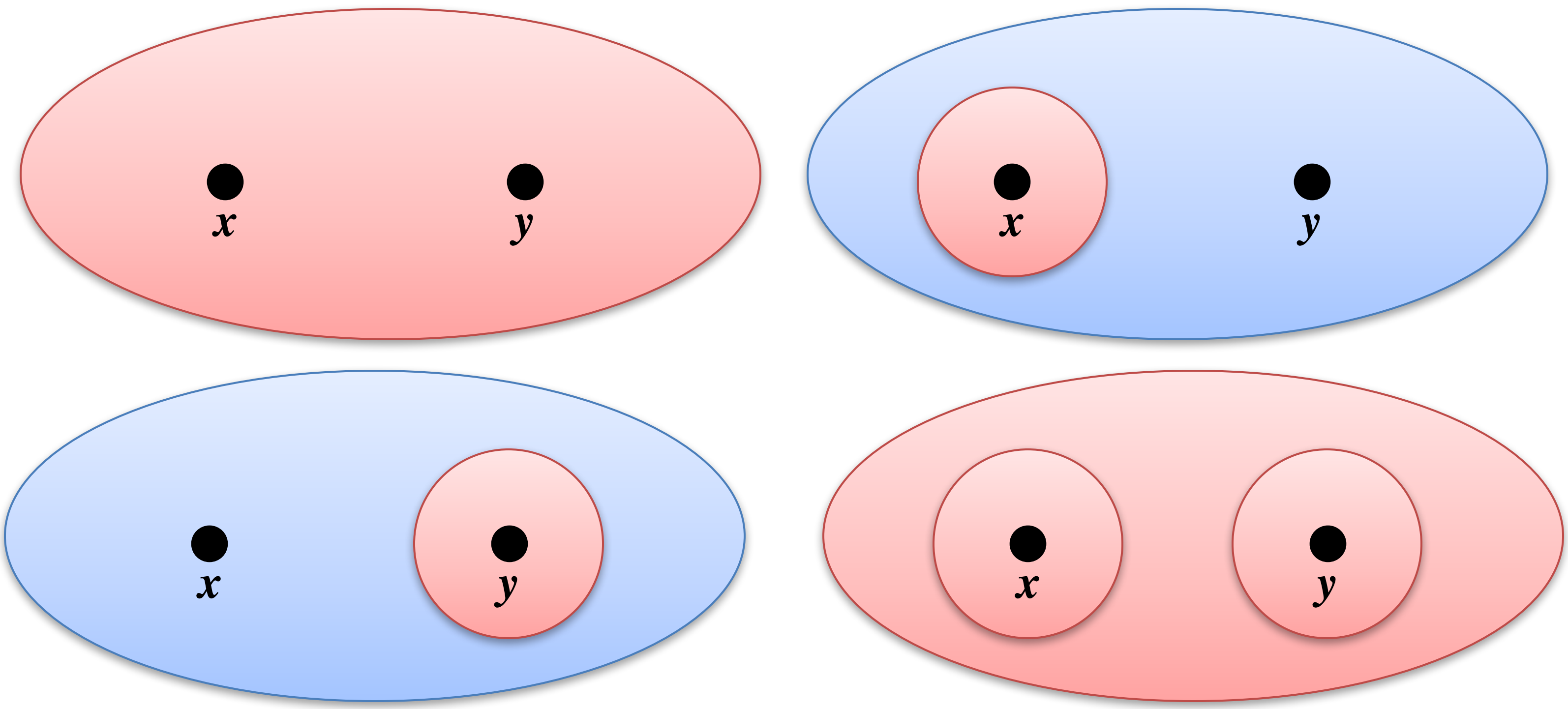
Definition 1.38: *Let* $\mathcal{X}$ *be a topological space with a basis* $\mathcal{B}$. *A subset* $\mathcal{O} \subseteq \mathcal{X}$ *is an* **open set** *if for each* $x \in \mathcal{O}$ *there is a neighborhood* $\mathcal{N} \in \mathcal{B}$ *such that* $x \in \mathcal{N}$ *and* $\mathcal{N} \subseteq \mathcal{O}$. *The set* $\mathcal{T}$ *of all open sets is a* **topology** *on the set* $\mathcal{X}$.

- Note that by definition, it is clear that any neighborhood is itself an open set, hence it is possible to define everything using only open sets, however the smaller collection of neighborhoods is usually easier to work with.

Open Sets and Neighborhoods

Theorem 1.9: *Let $\mathcal{X}$ be a topological space with a basis $\mathcal{B}$ and topology $\mathcal{T}$. A set $\mathcal{O} \subseteq \mathcal{X}$ is open (note that $\mathcal{O} \in \mathcal{T}$) if and only if $\mathcal{O}$ can be written as union of elements of $\mathcal{B}$ which are open sets.*

$\mathcal{T} = \{\emptyset, \mathcal{X}\}$ $\quad$ $\mathcal{T} = \{\emptyset, \{x\}, \mathcal{X}\}$



$\mathcal{T} = \{\emptyset, \{y\}, \mathcal{X}\}$ $\quad$ $\mathcal{T} = \{\emptyset, \{x\}, \{y\}, \mathcal{X}\}$

Schematic representation of all the topologies on a set with two elements, i.e. let $\boldsymbol{\mathcal{X}} = \{\boldsymbol{x}, \boldsymbol{y}\}$, all possible topologies defined on $\boldsymbol{\mathcal{X}}$ are $\boldsymbol{\mathcal{T}} = \{\emptyset, \boldsymbol{\mathcal{X}}\}, \boldsymbol{\mathcal{T}} = \{\emptyset, \{\boldsymbol{x}\}, \boldsymbol{\mathcal{X}}\}$, $\boldsymbol{\mathcal{T}} = \{\emptyset, \{\boldsymbol{y}\}, \boldsymbol{\mathcal{X}}\}$and $\boldsymbol{\mathcal{T}} = \{\emptyset, \{\boldsymbol{x}\}, \{\boldsymbol{y}\}, \boldsymbol{\mathcal{X}}\}$, note that the empty set $\emptyset$ must always be considered as an open set.

Open Sets and Neighborhoods

Definition 1.39: *If $\mathcal{C}$ is a subset of a topological space $\mathcal{X}$ with topology $\mathcal{T}$ then $\mathcal{C}$ is closed if $\mathcal{X} - \mathcal{C}$ is open.*

Theorem 1.10: *Let $\mathcal{X}$ be a topological space:*

(1) The empty set $\emptyset$ and $\mathcal{X}$ are closed sets.
(2) The intersection of any collection of closed sets in $\mathcal{X}$ is closed.
(3) The union of any finite collection of closed sets in $\mathcal{X}$ is closed.

If $\mathcal{X}$ is a topological space with a subset A, then A inherits a topology from $\mathcal{X}$ called the subspace topology defined and the relative topology.

Definition 1.40: *Let $\mathcal{X}$ be a topological space with topology $\mathcal{T}$ and let $A \subseteq \mathcal{X}$, a neighborhood of a point $x \in A$ relative to A is of the form $\mathcal{N} \cap A$ when $\mathcal{N}$ is a neighborhood of x in $\mathcal{X}$, the topology $\mathcal{T}_A$ generated by this basis is called the subspace topology on A induced by the topology $\mathcal{T}$ on $\mathcal{X}$.*

Theorem 1.11: *Let A be a subset of a topological space $\mathcal{X}$, the open sets relative to A are precisely the open sets of $\mathcal{X}$ intersected with A, i.e. B is open in A if and only if $B = A \cap O$ for some set O which is open in $\mathcal{X}$, furthermore B is closed in A if and only if $B = A \cap C$ for some set C which is closed in $\mathcal{X}$.*

Continuity, Connectedness and Compactness

Definition 1.41: *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be topological spaces, then a function* $f: \mathcal{X} \to \mathcal{Y}$ *is continuous if whenever a set* A *is open in* $\mathcal{Y}$, $f^{-1}(A)$ *is an open set in* $\mathcal{X}$, *if we let* $\mathcal{T}_{\mathcal{X}}$ *denote the topology on* $\mathcal{X}$ *and* $\mathcal{T}_{\mathcal{Y}}$ *the topology for* $\mathcal{Y}$, *this can be restated as:* f *is continuous if for every* $A \in \mathcal{T}_{\mathcal{Y}}$, *we have* $f^{-1}(A) \in \mathcal{T}_{\mathcal{X}}$.

Definition 1.42: *A topological space* $\mathcal{X}$ *is connected if* $\mathcal{X}$ *cannot be written as a union of two non-empty disjoint open sets.*

Theorem 1.12: *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be topological spaces, and a function* $f: \mathcal{X} \to \mathcal{Y}$ *be a continuous function onto* $\mathcal{Y}$, *if* $\mathcal{X}$ *is connected then* $\mathcal{Y}$ *is connected.*

Definition 1.43: *Let* A *be a subset of a topological space* $\mathcal{X}$, *an open cover of* A *is a collection* $\mathcal{O}$ *of open subsets of* $\mathcal{X}$ *such that* A *lies in the union of elements of* $\mathcal{O}$, *i.e.* $A \subseteq \bigcup_{O \in \mathcal{O}} O$.

Continuity, Connectedness and Compactness

A subcover of $\mathcal{O}$ is a subcollection $\mathcal{O}' \subseteq \mathcal{O}$, so that A lies in the union of the elements of $\mathcal{O}'$. A finite cover (or subcover) is a cover $\mathcal{O}$ consisting of finitely many sets.

Definition 1.44: *A topological space $\mathcal{X}$ is compact if every open cover of $\mathcal{X}$ has a finite subcover.*

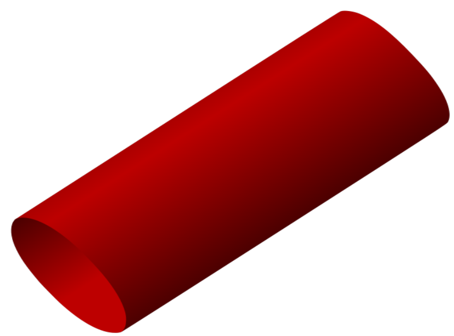
Theorem 1.13: *If $\mathcal{X}$ is a compact topological s[ace and $\mathbf{A}$ is a closed subset of $\mathcal{X}$, then $\mathbf{A}$ is compact.*

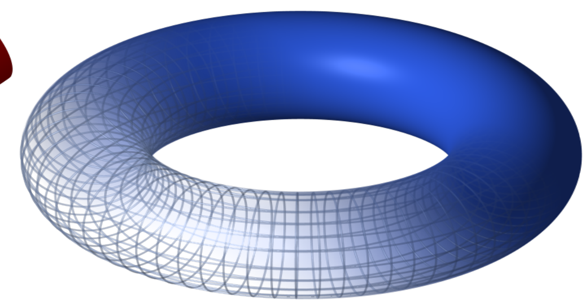
Theorem 1.14: *Let $\mathcal{X}$ be a compact topological space and $f:\mathcal{X} \to \mathcal{Y}$ be a continuous function from $\mathcal{X}$ onto a topological space $\mathcal{Y}$, then $\mathcal{Y}$ is compact.*

Product Spaces

- Now, we will see a way of creating new topological spaces from old ones.

Definition 1.45: *Let $\mathcal{X}$ and $\mathcal{Y}$ be any spaces, the (Cartesian) product of $\mathcal{X}$ and $\mathcal{Y}$ is the set of all ordered pairs (x, y) such that* $\mathcal{X} \times \mathcal{Y} = \{(x, y): x \in \mathcal{X} \text{ and } y \in \mathcal{Y}\}$.





Let I denote the unit interval $[0,1]$ and $\mathbb{S}^1$ the unit circle, then $I \times I$ is the unit square, $\mathbb{S}^1 \times I$ is a cylinder and $\mathbb{S}^1 \times \mathbb{S}^1$ is the skin of a doughnut or torus.

Theorem 1.15: *Let $\mathcal{X}$ and $\mathcal{Y}$ be connected topological spaces. The product $\mathcal{X} \times \mathcal{Y}$ is connected.*

Theorem 1.16: *Let $\mathcal{X}$ and $\mathcal{Y}$ be compact topological spaces. The product $\mathcal{X} \times \mathcal{Y}$ is compact .*

Agenda

- Overview
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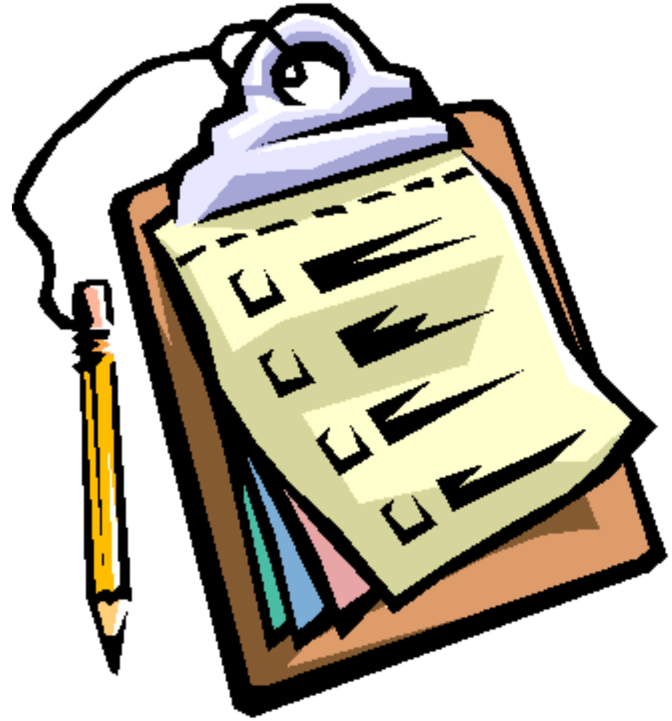


Using *combinatorial* approach to topology allows us to study spaces which are built from a uniform set of building blocks; hence we need to study the blocks themselves and how they are combined to form a topological space. The basic building blocks are called *cells* which are assembled into *complexes*. The cells are of varying dimensions however we will concentrate on spaces which are locally 2-dimensional.

Complexes

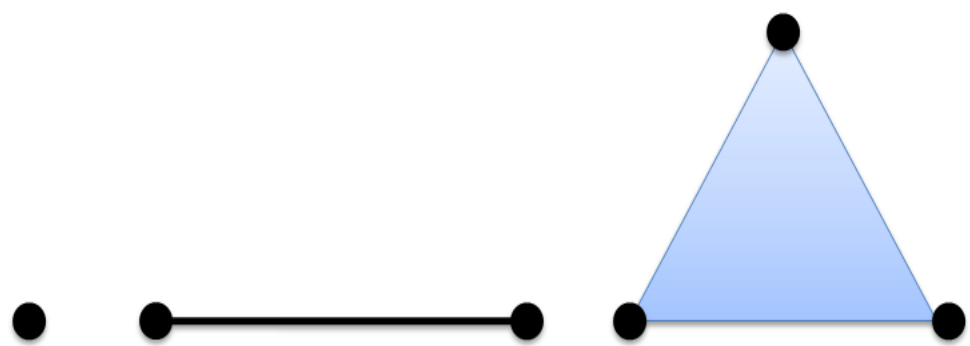
Definition 1.46: *An* **n-cell** *is a set whose interior is homeomorphic to the n-dimensional disc* $\mathcal{D}^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$ *with the additional property that its boundary or frontier must be divided into a finite number of lower-dimensional cells, called the* **faces** *of the n-cell. We write* $\sigma \prec \tau$ *if* σ *is a face of* τ.

(1) A 0-dimensional cell is a point $\mathbf{A}$.

(2) A 1-dimensional cell is a line segment $a = \mathbf{AB}$, *where* $\mathbf{A} \prec a$ *and* $\mathbf{B} \prec a$, *that is* $\mathbf{A}$ *and* $\mathbf{B}$ *are faces of the cell* a.

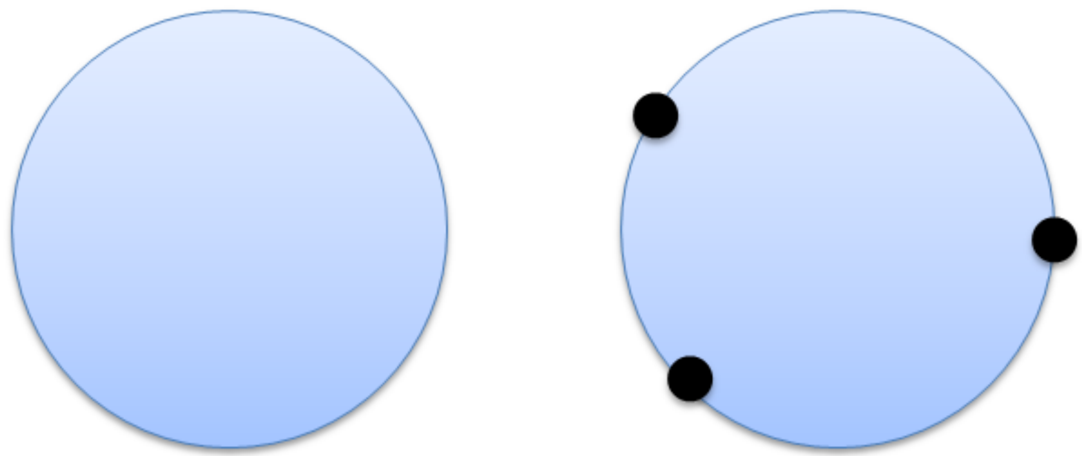
(3) A 2-dimensional cell is a polygon (often a triangle), such as $\sigma = \Delta\mathbf{ABC}$, *and then* $\mathbf{AB}, \mathbf{BC}, \mathbf{AC} \prec \sigma$, *note that* $\mathbf{A} \prec \mathbf{AB} \prec \sigma$, *hence* $\mathbf{A} \prec \sigma$.

(4) A 3-dimensional cell is a solid polyhedron (often a tetrahedron) with polygons, edges and vertices as faces.



Faces of an *n*-cell are lower-dimensional cells: the endpoints of a 1-cell or edge are 0-cells or points, the boundary of a 2-cell or polygon consists of edges (1-cells) and vertices (0-cells), these cells will be assembled together to form complexes.

Complexes



The space on the left is not considered to be a cell, it satistisfies the first condition of the definition where its interior is homemorphic (topoloigcally equivalent) to a 2-disc, however its boundary is a circle which is not a 1-cell, in order to be considered as a cell, its boundary should be divided into finite number of lower dimensional cells, i.e. edges (1-cells) and vertices (0-cells) as illustrated on the right.

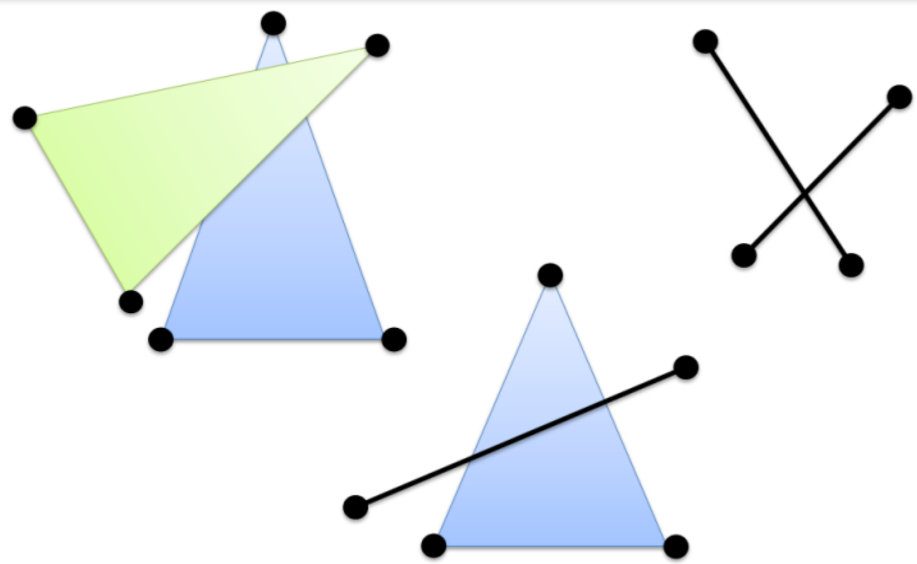
Complexes

Definition 1.47: *A complex* $\mathcal{K}$ *is a finite set of cells, i.e.* $\mathcal{K} = \cup\{\sigma : \sigma \text{ is a cell}\}$, *such that:*

(1) If σ *is a cell in* $\mathcal{K}$, *then all faces of* σ *are elements of* $\mathcal{K}$.

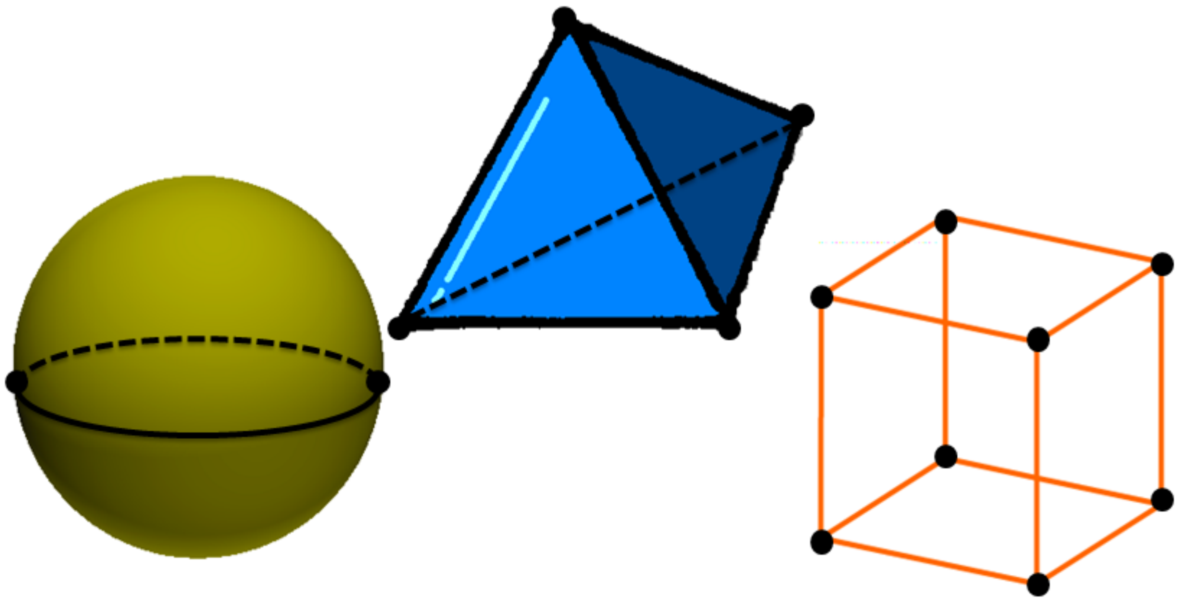
(2) If σ *and* τ *are cells in* $\mathcal{K}$, *then* $Int(\sigma) \cap Int(\tau) = \emptyset$.

The dimension of $\mathcal{K}$ *is the dimension of its highest-dimensional cell.*



- The second condition of the definition prohibits these intersections between cells. It is important to note that a complex is more than a set of points, since points are arranged into cells with various dimensions. In each case above the intersections are homeomorphic to cells however they are not among the cells of the complex

Complexes



- A topological object may be represented by many complexes, we mentioned before that these shapes are topologically equivalent, hence they represented the same topological space but with different complex structures.

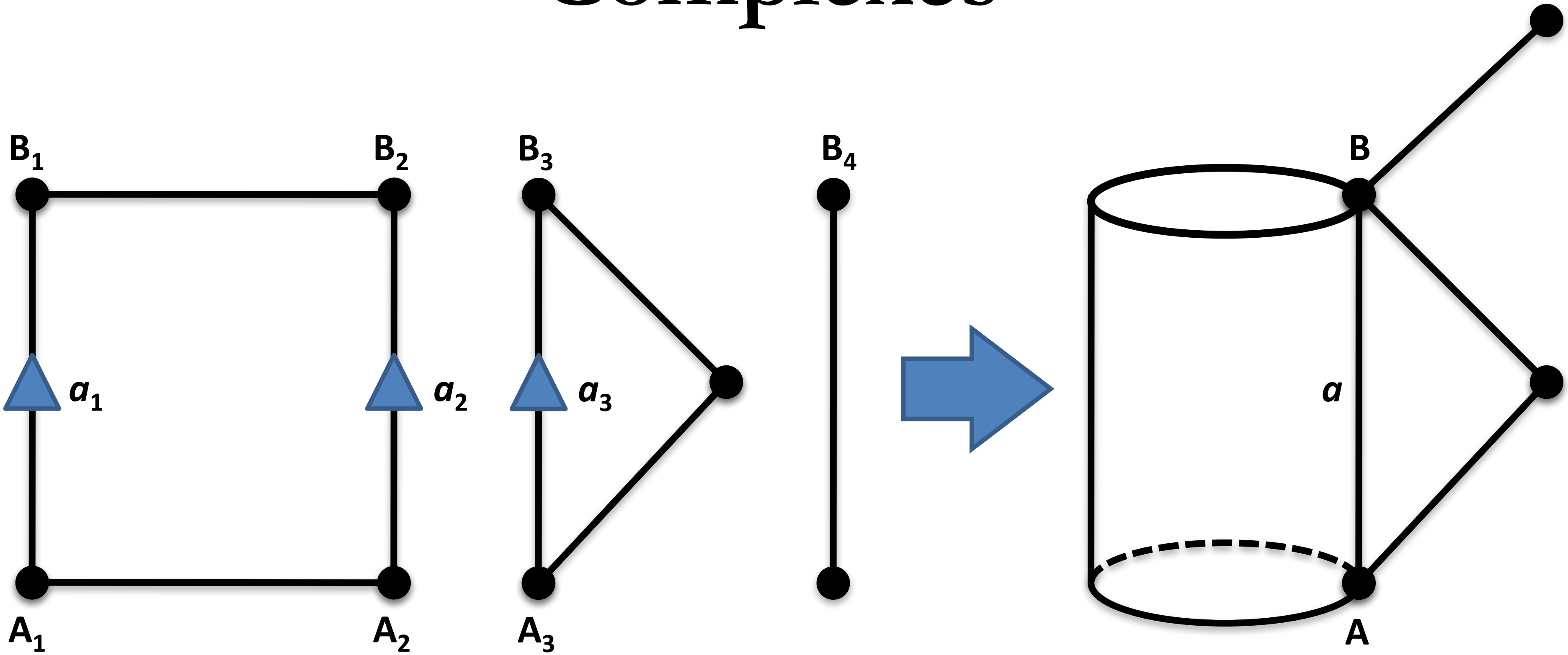
Complexes

Definition 1.48: *Let* $\mathcal{K}$ *be a complex. The set of all points in the cells of* $\mathcal{K}$ *is defined as* $|\mathcal{K}| = \{x : x \in \sigma \in \mathcal{K}, \sigma \text{ is a cell in } \mathcal{K}\}$ *is the* **space** *underlying the complex* $\mathcal{K}$*, or the* **realization** *of* $\mathcal{K}$.

The main difference between a space and a complex is that a complex $\mathcal{K}$ is a set of cells, it is a layered structure build up of cells of various dimensions, while a space $|\mathcal{K}|$ is a set of points.

Definition 1.49: *Let* $\mathcal{K}$ *be a complex. The k-skeleton of* $\mathcal{K}$ *is* $\mathcal{K}_k = \{k - \text{cells of } \mathcal{K}\}$*, that is k-skeleton is the set of cells in* $\mathcal{K}$ *having a dimension of k. Noting that* $\mathcal{K}_k$ *is a k-complex and* $\mathcal{K} = \cup_{k=1}^{n} \mathcal{K}_k$ *where* $n = \dim(\mathcal{K})$*, which is the dimension of its highest-dimensional cell.*

Complexes



The complex $\mathcal{K}$ is formed from two polygons (the square and the triangle) and the extra edge, by gluing/identifying the edges labeled a_1, a_2 and a_3 to form one new edge labeled 'a' and identifying the vertices A_1, A_2 and A_3 to the vertex A in $\mathcal{K}$, and B_1, B_2, B_3 and B_4 to from the vertex B in $\mathcal{K}$. The arrows can be used instead of labeling the vertices to specify a direction/orientation for the edges which must be respected when the gluing is done. Note that a single point on the edge 'a' in $\mathcal{K}$ correspond to three different points on the two polygons, whereas the vertex A correspond to three vertices while the vertex B correspond to four vertices.

Complexes

- 2-complexes have three types of points:
  - Points which lie in the interior of one of the 2-cells or polygons.
  - Points which lie in the interior of one of the edges.
  - Vertex points.

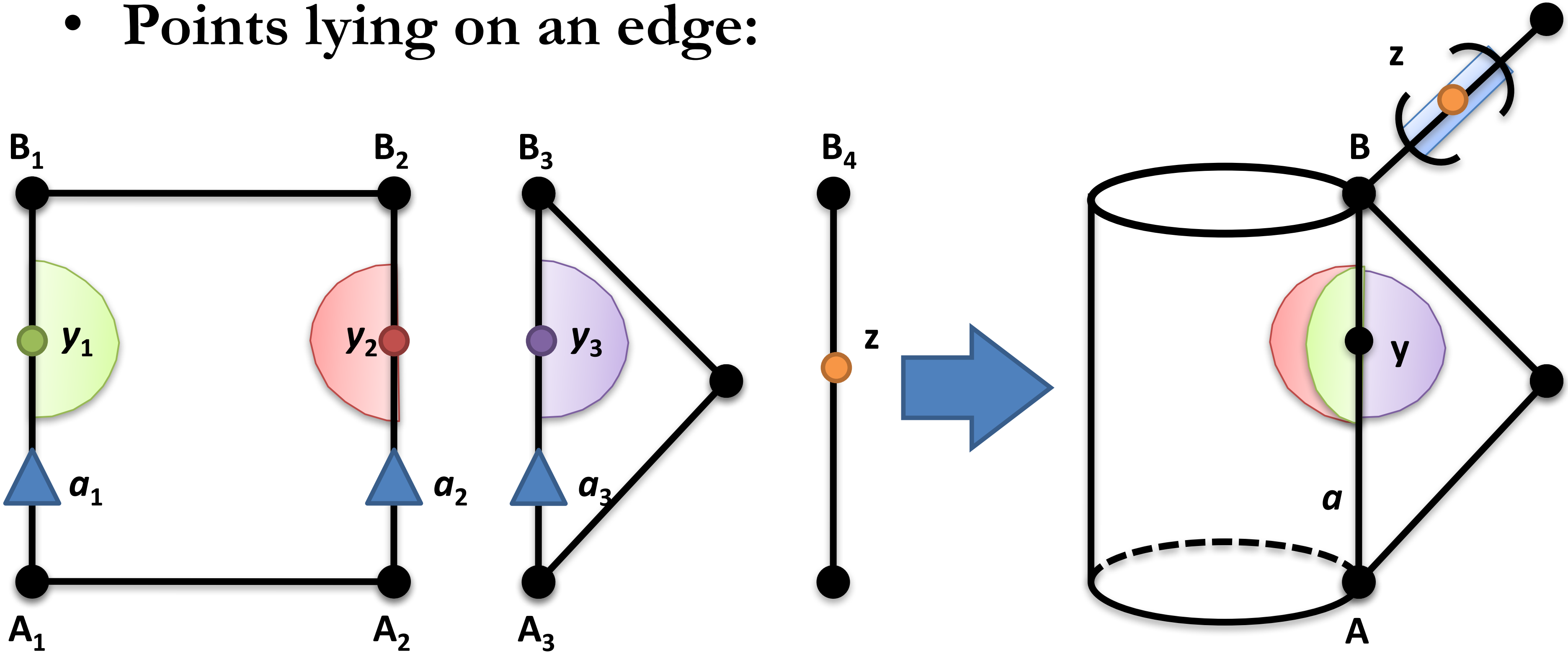
Complexes

Now, we will defined the **neighborhoods** for each type of points starting with a set of polygons $\mathcal{P} = \{P_i\}$.

Points lying inside a 2-cell: if a point x lies in the interior of a polygon P_i then one can define the neighborhood of x to be any disc totally enclosed in the interior of P_i.

Complexes

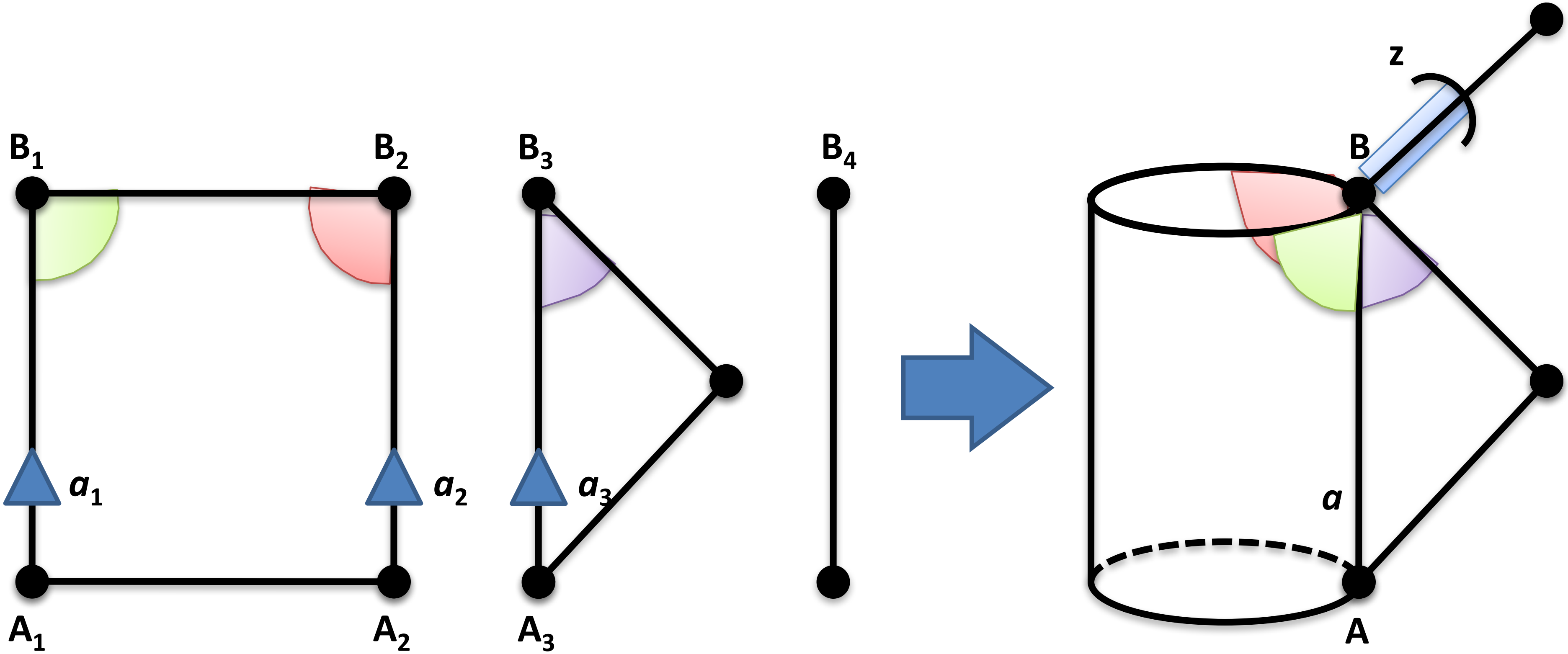
- **Points lying on an edge:**



- Possible neighborhoods of points lying on an edge, z lies on an edge which is not a part of the boundary of any 2-cell, while y lies on an edge of the complex.

Complexes

- **Vertex points:**

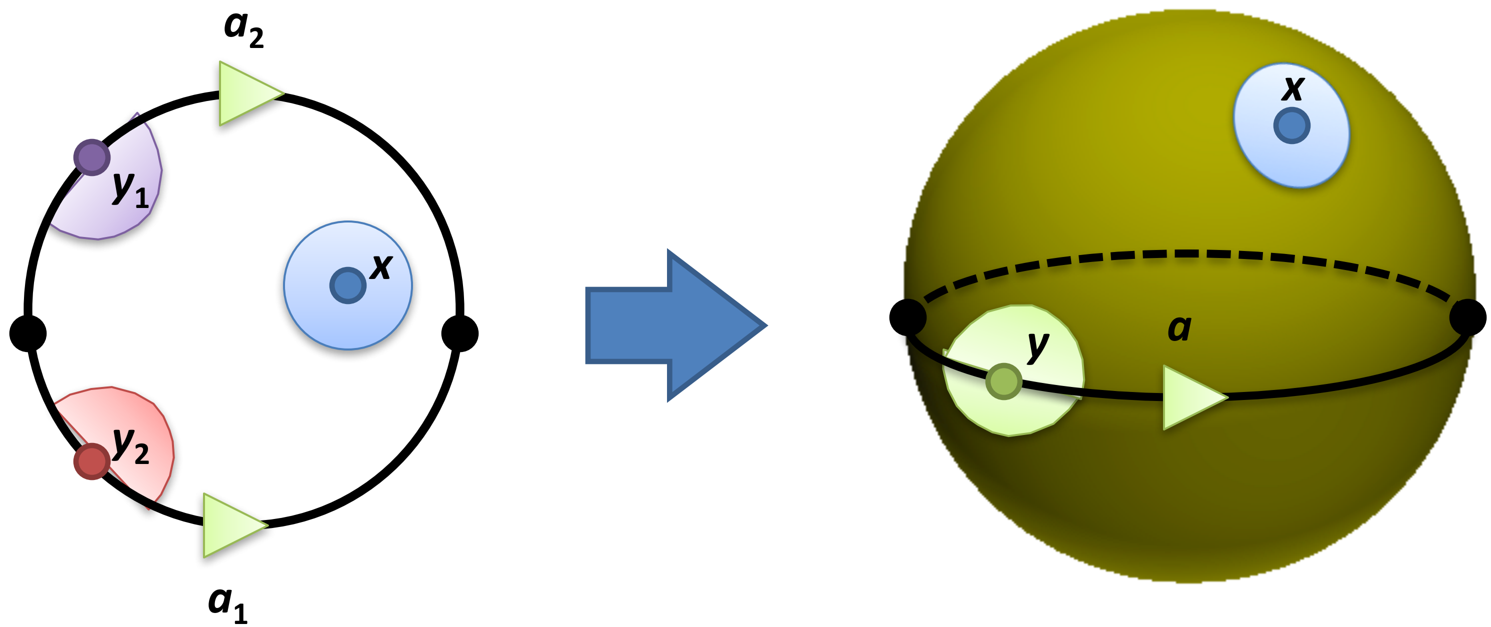


Manifolds and Surfaces

Definition 1.50: *An n-dimensional* **manifold** *is a topological space such that every point has a neighborhood topologically equivalent to an n-dimensional open disc with center* $\mathbf{x}$ *and radius* r, *i.e.* $\mathcal{D}^n(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{y}\| < r\}$. *We further require that any two distinct points has disjoint neighborhoods. A 2-manifold is often called a* **surface.**

Example: The sphere, denoted by $\mathbb{S}^2$, is a surface, even though it exists in a 3-dimensional spaces (this emphasizes the difference between intrinsic and extrinsic properties, where intrinsic properties have to do with the object itself, in contrast to extrinsic properties which describe how the object is embedded in the surrounding space). If one considers a point $\mathbf{x} \in \mathbb{S}^2$ as a point in $\mathbb{R}^3$, it will have a neighborhood that looks like a ball. However as a point on the sphere, with the relative topology, $\mathbf{x}$ has neighborhoods of the form: $\mathcal{N} = \mathcal{D}^n(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^3 : \|\mathbf{x} - \mathbf{y}\| < r \textit{ and } \mathbf{y} \in \mathbb{S}^2\}$, these neighborhoods look like 2-dimensional discs which have been warped a bit. See the following figure.

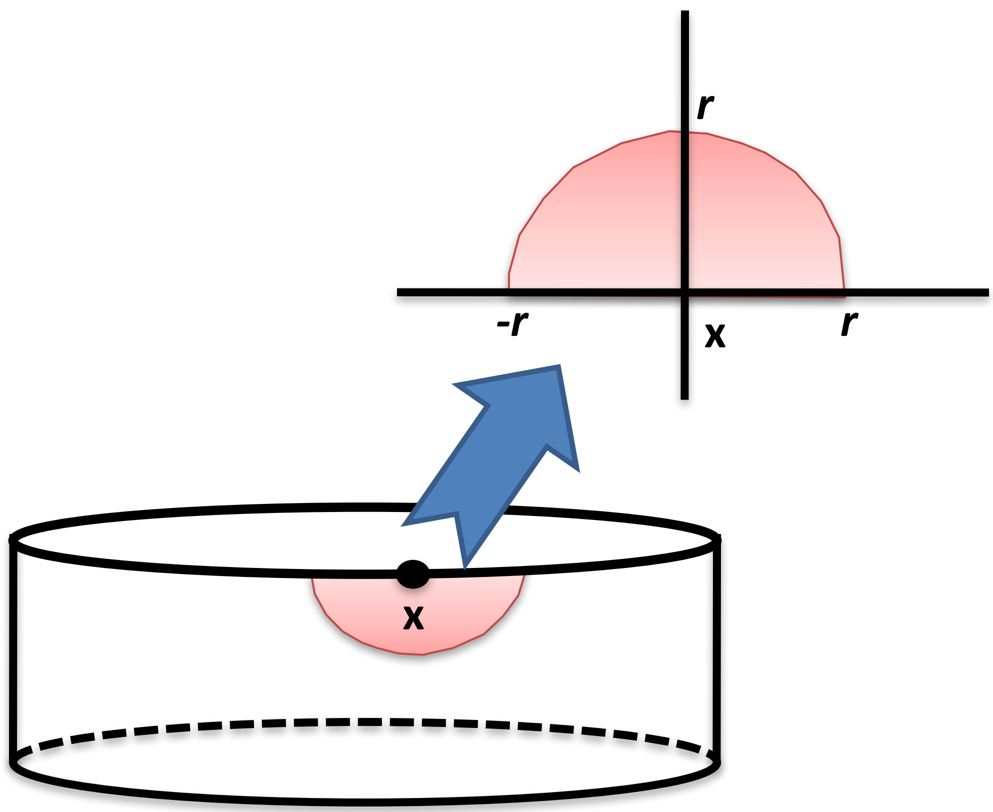
Manifolds and Surfaces



The sphere is represented by a disc with a zipper. On this planar diagram for the sphere, neighborhoods of interior points like x are open discs totally enclosed in the 2-cell disc which form the sphere, while points along the edge such as y has neighborhoods are half-discs, such that when the edges of the planar diagram are glued together, these half-discs are also glued to form a neighborhood of y topologically equivalent to a disc.

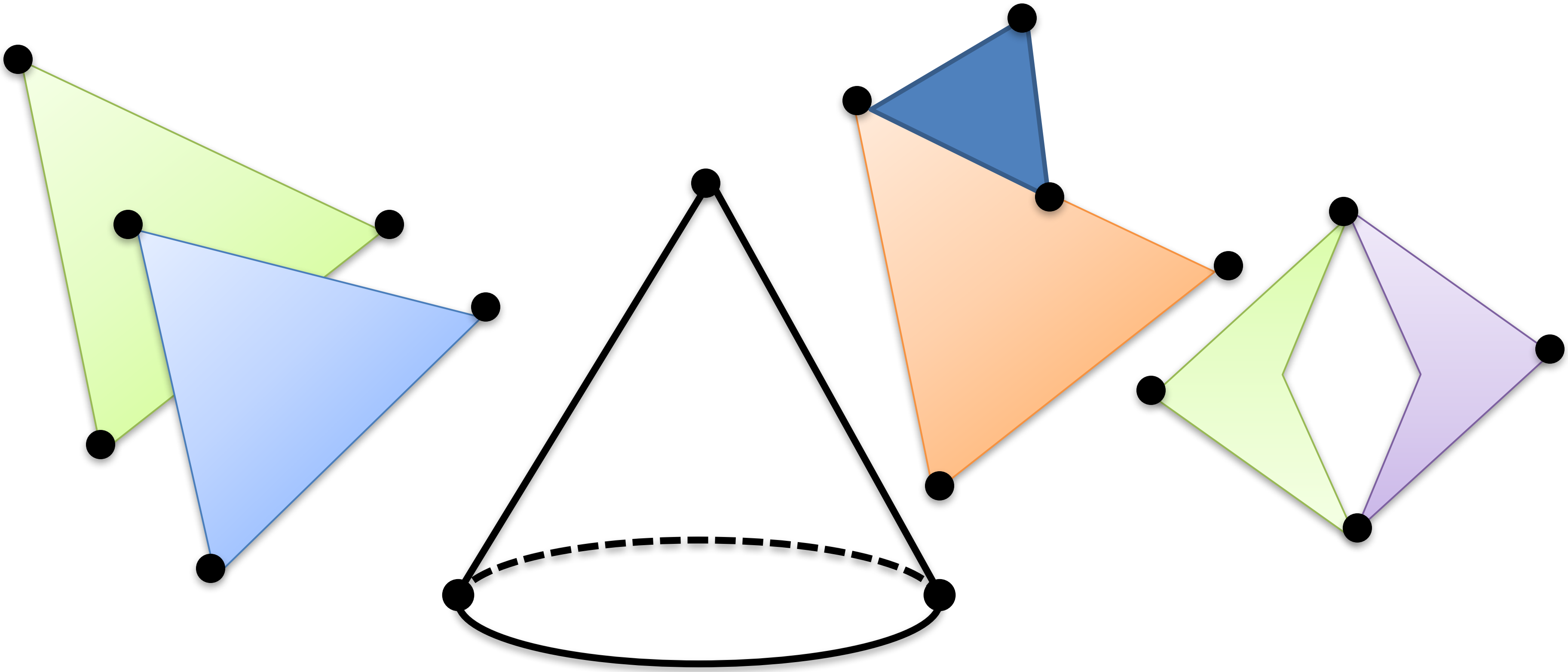
Manifolds and Surfaces

Definition 1.51: *An n-***manifold with boundary** *is a topological space such that every point has a neighborhood topologically equivalent to either an n-dimensional open disc, i.e.* $\mathcal{D}^n = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| < r\}$, *or the half-disc* $\mathcal{D}_+^n = \{\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \|\mathbf{x}\| < r \text{ and } x_n \geq 0\}$. *Points with half-disc neighborhoods are called* **boundary points**, *where the homomorphism must send a boundary point to a point with* $x_n = 0$. *The edge of n-manifold is an (n-1)-manifold.*



Triangulations

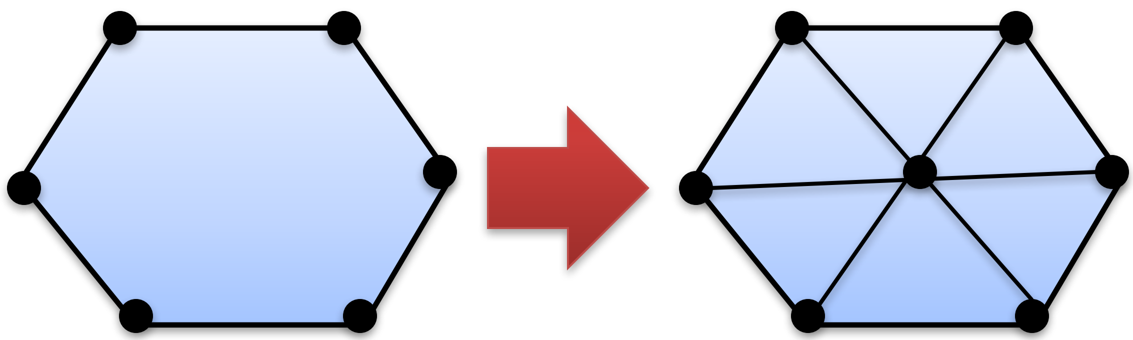
Definition 1.52: *A locally 2-dimensional topological space* $\mathcal{X}$ *is* **triangulable** *if a 2-complex structure* $\mathcal{K}$ *can be found with* $\mathcal{X} = |\mathcal{K}|$*, and* $\mathcal{K}$ *has only triangular cells satisfying the additional condition that any two triangles are identified along a single edge or at a single vertex or are disjoint. A triangulated complex* $\mathcal{K}$ *is called a* **simplicial** *complex or a triangulation on* $\mathcal{X}$. *A cell of a simplicial complex is called a* **simplex.**



- These are valid cell complexes with only triangular 2-cells, but are not triangulations

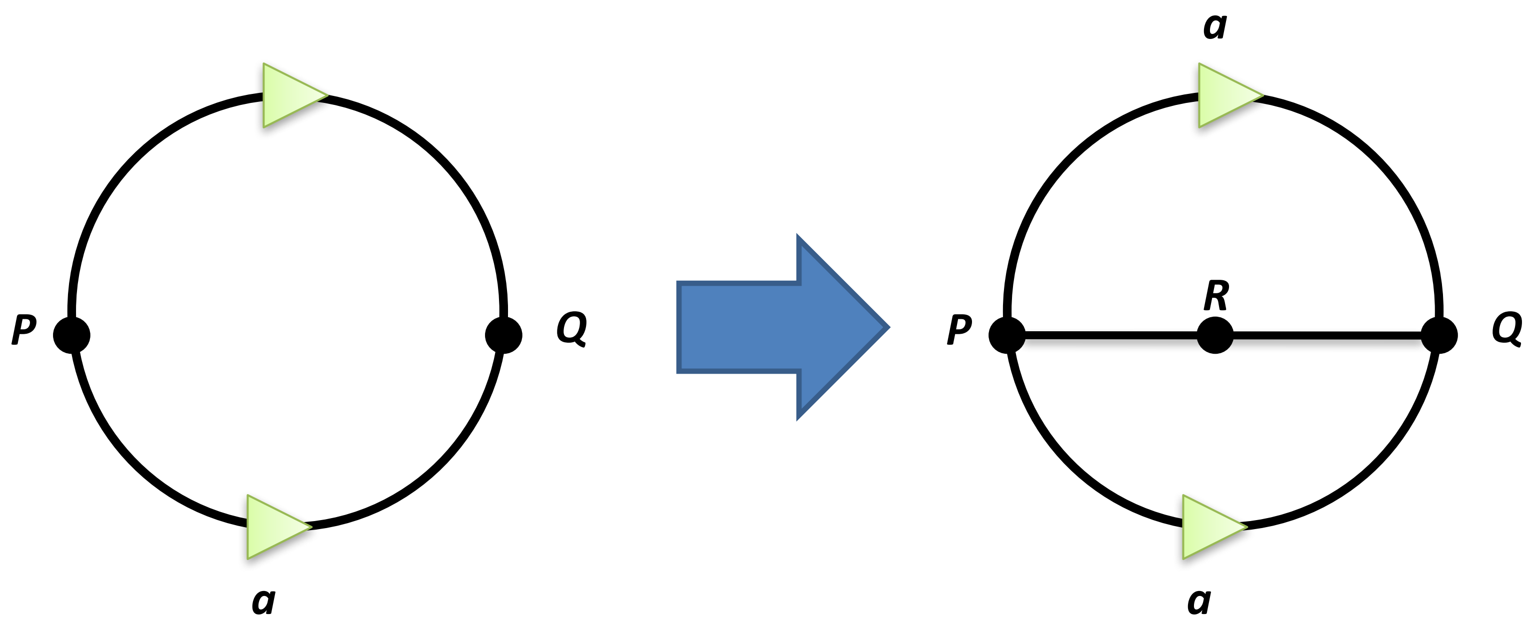
Triangulations

- We often want to triangulate a given complex. Each face of a 2-complex is a polygon which can be easily divided into triangles by introducing a new vertex in the interior of the polygon then connecting this new vertex to each of the vertices on the boundary of the polygon.



Triangulations

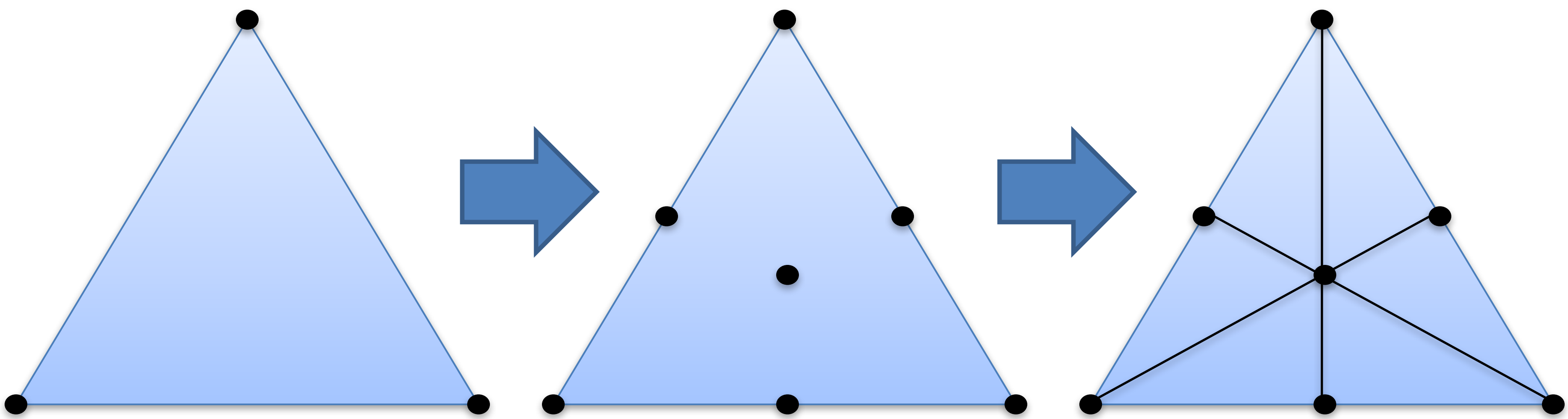
- However, it should be noted that although this process gives a method for dividing any 2-complex into triangles, it does not always give a triangulation satisfying Definition 1.52.



- The planar diagram for the sphere is divided into triangles, but his is not a triangulation since this complex has two different triangles labeled PQR, hence it cannot be simplicial. The triangles must be further subdivided

Triangulations

Definition 1.53: *Let $\mathcal{K}$ be a 2-complex with triangular 2-cells. A new complex $\mathcal{K}'$ called the* **barycentric subdivision** *of $\mathcal{K}$ is formed by introducing an new vertex at the center of each triangle and a new vertex at the midpoint of each edge and drawing edges from the center vertex to each of the new midpoint vertices and to the original vertices. In general, this is described as creating a new vertex v_σ in the center of every cell $\sigma \in \mathcal{K}$, including any vertex P when we define $v_P = P$, and add a connecting cell from v_σ to v_τ whenever $\sigma \prec \tau$.*



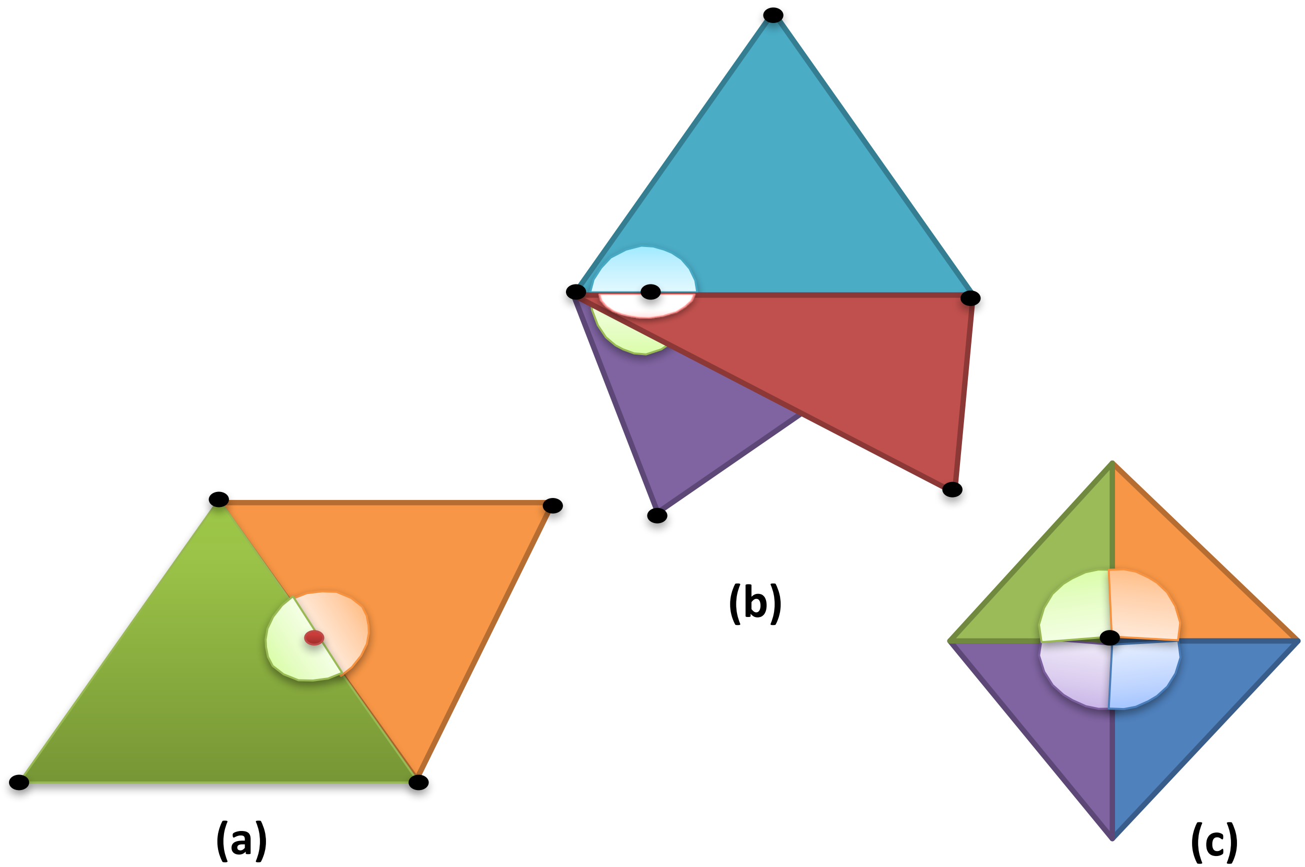
Triangulations

Definition 1.54: *A triangulated surface (without boundary) is a simplicial 2-complex such that:*

1) *each edge is identified/glued to exactly one other edge;*
2) *the triangles meeting at a vertex can be labeled* $T_1, T_2, \ldots, T_n$ *with adjacent triangles in this sequence identified/glued along an edge and* T_n *is glued to* T_1 *along an edge.*

- The first condition guarantees points on an edge belong to exactly two triangles, and so a disc-like neighborhood exists for each point on the edge resulting from gluing two half-discs together, one from each triangle. While the second condition ensures that a neighborhood at a vertex looks like a disc formed from gluing *n*-triangles which share this vertex.

Triangulations



(a) an edge point on a surface has a disc-like neighborhood,

(b) an edge point on a complex which is not a surface,

(c) A vertex point on a surface has also a disc-like neighborhood

Triangulations

Theorem 1.17: *A surface is compact if and only if any triangulation uses a finite number of triangles.*

Theorem 1.18: *A surface is connected if and only if a triangulation can be arranged in order* $T_1, T_2, \ldots, T_n$ *with each triangle having at least one edge identified/glued to an edge of a triangle listed earlier.*

- A triangulated surface with boundary is a topological space obtained from a set of triangles with edges and vertices identified to satisfy Definition 1.52, except some edges will not be identified. These unmatched edges form the boundary of the surface.

Definition 1.55: *A triangulated surface with boundary is a topological space with a simplicial 2-complex such that;*

1) *Each edge is identified to at most one other edge.*
2) *The triangles meeting at a vertex can be labeled* $T_1, T_2, \ldots, T_n$ *with adjacent triangles in this sequence identified along an edge and* T_n *either glued to* T_1 *along an edge, or* T_n *and* T_1 *each have one edge on the boundary.*
3) *No edge not on the boundary can have both vertices on the boundary.*

- The third condition is added so that it is possible to clearly identify the vertices and edges of the boundary. If vertices **A** and **B** are on the boundary, then the edge **AB** must also lie on the boundary. In order to satisfy this condition, we may have to alter the simplicial complex by dividing some of the triangles and edges into smaller pieces by using the barycentric subdivision to obtain a new triangulation. In practice, however, it is more efficient to only subdivide edges and triangles as necessary, thus minimizing the number of new triangles.

Reference

- *Topology of Surfaces* by L. C. Kinsey (Springer, 1993)

Thank You